CSCE 619-600 Networks and Distributed Processing Spring 2017

Review of Probability II

Dmitri Loguinov
Texas A&M University

January 24, 2017

Agenda

- Memoryless property
- Gas station example
- Joint distributions
  - Continuous
  - Discrete

Memoryless Distributions

- We can now demonstrate how exponential distributions "forget" memory
- Suppose inter-bus delays are exponential and we know that the last bus left t time units ago
  - What is the distribution of the remaining wait time?
- Define X to be the inter-bus delay and $W_t = X - t$ to be the wait time conditioned on $X > t$
  - Then we have:

$$\begin{aligned} P(W_t > x) &= P(X - t > x | X > t) \\ &= P(X > x + t | X > t) \end{aligned}$$

Memoryless Distributions 2

- Rewriting:

$$P(W_t > x) = \frac{P(X > x + t, X > t)}{P(X > t)}$$
$$= \frac{F^c(t + x)}{F^c(t)}$$

- Now substituting the tail of the exponential distribution, we have complete independence of t:

$$P(W_t > x) = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$

Memoryless Distributions 3

- Distribution of delay to next bus is independent of t
  - In fact, W_t has the same exponential distribution as X
  - Explains why the wait was always 20 minutes in the previous lecture
- Definition
  - A distribution is called *memoryless* if and only if

$$P(X - t > x | X > t) = F^c(x)$$

- This means that the current "age" of X does not affect its remaining life

Memoryless Distributions 4

- The last equation can be re-written as:

$$\frac{F^c(t+x)}{F^c(t)} = F^c(x)$$

- We thus have an equivalent definition of memoryless distributions:

$$F^c(t+x) = F^c(x)F^c(t)$$

- Exercise: show that the exponential distribution is the only memoryless distribution

Practice

- Driver decisions
  - Your gas tank holds 20 gallons, but fuel gauge is broken
  - You drive around and all of a sudden see a good deal
  - If filling up X gallons and $X < 10$, you pay X^2 dollars, otherwise $1.5X$ dollars; but have to fill the tank to the max
  - You only have $20 on you
  - What is the likelihood of successful fill-up?
- Suppose Y is the cash due if you go for it

$$
\begin{aligned}
P(Y < 20) &= P(Y < 20 | X < 10) P(X < 10) \\
&+ P(Y < 20 | X \geq 10) P(X \geq 10)
\end{aligned}
$$

Practice 2

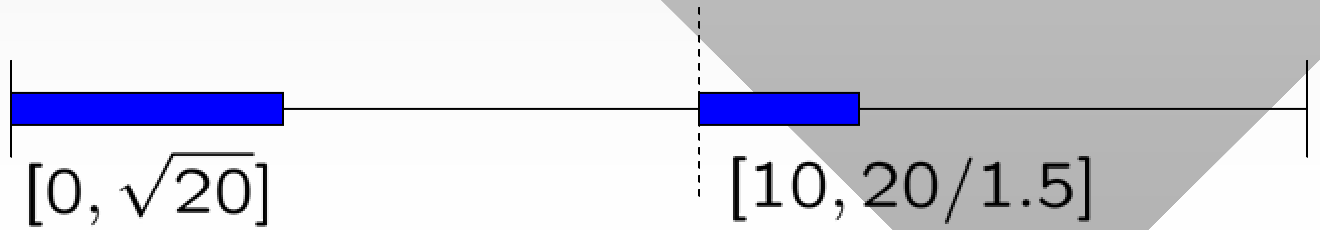
- Assuming a uniform distribution for X

$$
\begin{aligned}
P(Y < 20) &= P(0 \le X < \sqrt{20} | X < 10)\frac{1}{2} \\
&+ P(10 \le X < 20/1.5 | X \ge 10)\frac{1}{2}
\end{aligned}
$$

- This can be computed as

$$
P(Y < 20) = \frac{\sqrt{5} + 5/3}{10} \approx 0.3903\cdots
$$

- Graphical explanation only valid when X is uniform



Joint Distributions

- It is common that your model involves multiple random variables
  - Furthermore, sometimes they are dependent
- Example
  - Suppose you randomly generate a uniform number in $[0,10]$ and your friend generates one from the exponential distribution with mean 5
  - What's the probability that your number is greater?
- To deal with such cases methodically, we next define joint distributions

Joint Distributions 2

- Assume X_1, X_2, …, X_n are random variables
- Definition: their joint distribution function (CDF) is:

$$F(x_1, \cdots, x_n) = P(X_1 \le x_1, \cdots, X_n \le x_n)$$

- If X_i is drawn from $F_i(x)$, then each F_i is called a *marginal* distribution:

$$P(X_i \le x_i) = F_i(x_i) = F(\infty, \cdots, x_i, \cdots, \infty)$$

- Example: X is uniform in [0, 1], $Y = X^2$
  - Obtain their joint CDF

$$P(X \le x, Y \le y) = \min(x, \sqrt{y})$$

Joint Distributions 3

- The joint density function can be obtained by differentiating $F(x_1,\ldots,x_n)$
  - For two variables, we have:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

- Define R to be some 2D region and vector $\mathbf{x} = (x,y)$
  - Then, the probability that random point (X,Y) falls into R is the integral of joint density over R

$$P((X,Y) \in R) = \int_R f(\mathbf{x}) d\mathbf{x}$$

  - for example:

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x,y) dy dx$$

Joint Distributions 4

- Theorem: If X and Y are independent, their joint density and joint CDF are products of marginal densities and marginal CDFs, respectively
  - Proof: First CDF

$$F(x,y) = P(X \leq x, Y \leq y) = F_X(x)F_Y(y)$$

  - and then density:

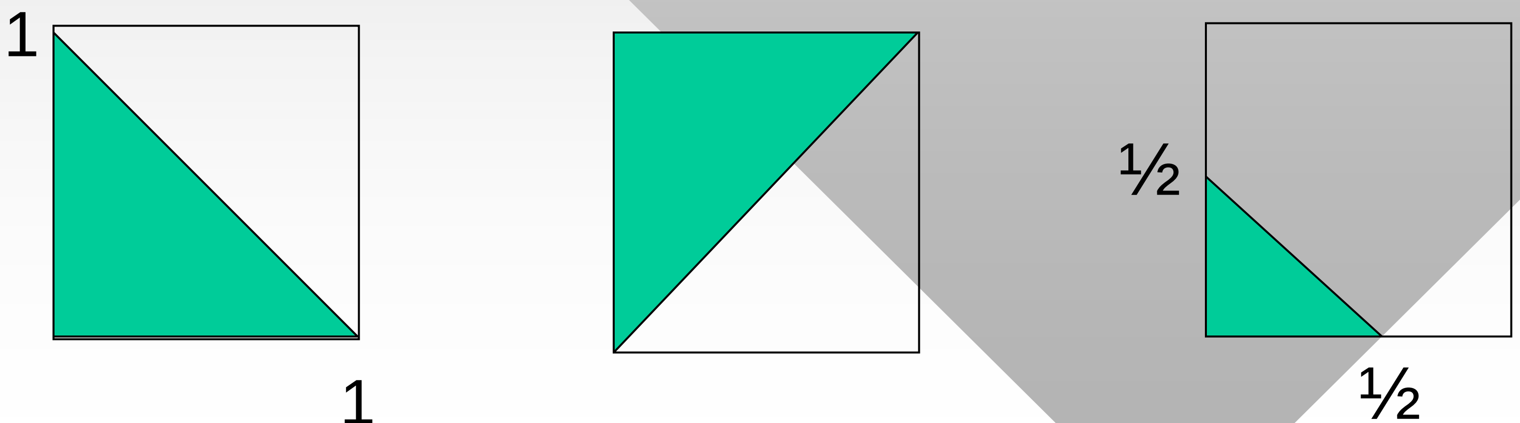
$$f(x,y) = \frac{\partial^2 [F_X(x)F_Y(y)]}{\partial x \partial y} = \frac{\partial [f_X(x)F_Y(y)]}{\partial y} = f_X(x)f_Y(y)$$

- Discrete distributions have a joint PMF defined as:

$$p(x,y) = P(X = x, Y = y)$$

Joint Distributions 5

- Example
  - Suppose X and Y are uniform in $[0,1]$ and independent
  - Define $Z = X+Y$
  - Derive $P(Z < 1)$, $P(X < Y)$ and $P(Z < ½)$
- First we need to deduce the integration region
  - Work this out



Joint Distributions 6

- Next we integrate the joint density
  - Notice that $f(x,y) = 1 \cdot 1 = 1$

$$P(Z < 1) = \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx = 1/2$$

$$P(X < Y) = \int_0^1 \int_x^1 f(x,y)\,dy\,dx = 1/2$$

$$P(Z < 1/2) = \int_0^{1/2} \int_0^{1/2-x} f(x,y)\,dy\,dx = 1/8$$

- If the variables are uniform, graphical solution is possible as well

Joint Distributions 7

- We can also derive $P(X < Y)$ simpler
  - Using independence:

$$P(X<Y)=\int_{-\infty}^{\infty}\int_{x}^{\infty} f_X(x) f_Y(y)dydx$$

  - Then:

$$P(X<Y)=\int_{-\infty}^{\infty}\left[f_X(x)\int_{x}^{\infty} f_Y(y)dy\right]dx$$

$$P(X<Y)=\int_{-\infty}^{\infty} P(Y>x)f_X(x)dx$$

- Also can be written as:

$$P(X<Y)=\int_{-\infty}^{\infty} P(X<y)f_Y(y)dy$$

Joint Distributions 8

- A more general result is possible
  - Continuous Y:

$$P(A) = \int_{-\infty}^{\infty} P(A|Y = y) f_Y(y) dy$$

  - Discrete Y with outcomes y_1, y_2, ... :

$$P(A) = \sum_{i=1}^{\infty} P(A|Y = y_i) P(Y = y_i)$$

- The discrete version was shown last time (since events $\{Y = y_i\}$ form a partition of Ω)

Joint Distributions 9

$$P(A) = \int_{-\infty}^{\infty} P(A|Y=y) f_Y(y) dy$$

- Example
  - As before, suppose you draw a number randomly from $[0,s]$ and your friend from an exponential distribution with mean $s\,/\,2$
  - Who will win more often and with what probability?
- Call X the first number and Y the second one
  - Since the density of X is simpler, we condition on X:

$$f_X(x) = 1/s, F_Y(x) = 1 - e^{-2x/s}$$

  - which leads to the same result for all s

$$P(X > Y) = \int_0^s F_Y(x) f_X(x) dx = \frac{1 + e^{-2}}{2} = 0.567\cdots$$