CSCE 619-600
Networks and Distributed Processing
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Review of Probability III

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Agenda

- Expectations
  - Of random variables
  - Of functions
- Examples

Expectations

- Expected value of a random variable
  - Also called the mean
  - Intuitive definition: if we generate infinitely many samples of X, their average will be $E[X]$
- Definition: expectation $E[X]$ is given by:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

- For a discrete variable with outcomes x_1, x_2, ...

$$E[X] = \sum_{i=1}^{\infty} x_i P(X = x_i)$$

Expectations 2

- For non-negative variables, there is a very useful alternative formula for $E[X]$
  - Recall integration by parts

$$\int_0^\infty u dv = uv\Big|_0^\infty - \int_0^\infty v du$$

- From which

$$\begin{aligned}\int_0^\infty x f(x) dx &= \int_0^\infty x dF(x) = -\int_0^\infty x dF^c(x) \\ &= -xF^c(x)\Big|_0^\infty + \int_0^\infty F^c(x) dx \\ &= \int_0^\infty F^c(x) dx\end{aligned}$$

Expectations 3

- Thus, for non-negative continuous variables:

$$E[X] = \int_0^\infty F^c(x)dx$$

  - Integration of the tail yields the mean
- Similarly, for non-negative integer-valued variables:

$$E[X] = \sum_{i=0}^\infty P(X > i)$$

- Example: compute the mean of exponential X

$$E[X] = \int_0^\infty e^{-\lambda x} dx = \left.\frac{e^{-\lambda x}}{-\lambda}\right|_0^\infty = \frac{1}{\lambda}$$

rate (λ)

Expectations 4

- Example: compute the expectation of the Pareto distribution, i.e., $F(x) = 1 - (1+x/\beta)^{-\alpha}$ for $x \in [0, \infty)$

$$E[X] = \int_0^\infty (1+x/\beta)^{-\alpha} dx = \frac{\beta}{1-\alpha} (1 + x/\beta)^{1-\alpha} \Big|_0^\infty$$

  - which reduces to

$$E[X] = \frac{\beta}{\alpha - 1}$$

- Also notice that if $\alpha < 1$, then $E[X] = \infty$
- Example: compute the mean wait time $E[W_t]$

Expectations 5

$$E[XY] = \int\int xyf(x,y)dydx$$

- Expectations are linear:

$$E[cX] = cE[X], \quad E[X+Y] = E[X] + E[Y]$$

  - For independent X,Y: $E[XY] = E[X]E[Y]$
- Definition: for arbitrary functions $u(.)$

$$E[u(X)] = \int_{-\infty}^{\infty} u(x)f(x)dx$$

- Example: compute $E[e^X]$ for uniform X in $[0, s]$

$$E[e^X] = \frac{1}{s}\int_0^s e^x dx = \frac{e^s - 1}{s}$$

Expectations 6

- What the last example means is that if you generate a sequence of random numbers $x_1,\ldots,x_n$ using the distribution of X, then the mean

$$\lim_{n\to\infty} \frac{e^{x_1} + \cdots + e^{x_n}}{n} = \frac{e^s - 1}{s}$$

- Simulation: $s = 5$
  - I generated $10{,}000$ random numbers in $[0,5]$
  - Then averaged e^{x_i} and obtained 29.35
  - The analytical result predicts 29.48
  - Notice that this does not equal $e^{E[X]} = 12.18$

Examples

- Note: unless specified otherwise, all variables in various problems are independent
- Compute $P(3X < Y)$ for X uniform in $[-2, 3]$ and Y exponential with mean $1/\lambda$
- Assume X is discrete with PMF $p_j = P(X = j)$, where $p_1 = 0.3$, $p_{34} = 0.5$, and $p_{90} = 0.2$
  - Compute the distribution of $Y = e^X$
- Compute $P(X>2 \mid X<3)$ for exponential X with λ
- How about $P(X+Y<10 \mid X<3)$ for X uniform in $[0, 5]$ and Y exponential with rate λ
- Compute $E[3X^2Y]$ for X uniform in $[0,1]$ and Y Pareto