CSCE 619-600
Networks and Distributed Processing
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Markov Chains II

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Agenda

- Markov chains
  - Definitions
  - Transitional probabilities
- Packet loss example
- Stationary distribution
  - Under convergence assumption

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$$p_{ij} = P(X_{n+1} = j | X_n = i)$$

- Matrix $P = (p_{ij})$ is called the one-step transition probability matrix
- Example: determine P for X_n being the number of heads in n coin-flips

$$p_{ii} = P(X_{n+1} = i | X_n = i) = 1 - p$$

$$p_{i,i+1} = P(X_{n+1} = i + 1 | X_n = i) = p$$

$$P = \begin{pmatrix} 1-p & p & & 0 \\ & 1-p & p & \\ & & \cdots & \\ 0 & & & \cdots \end{pmatrix}$$

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- Notice that the sum of each row in P equals 1
  - Any matrix with this property is called *stochastic*
  - Reason: the summation of transition probabilities out of any state i must be 1 (i.e., $\Sigma_j \, p_{ij} = 1$)
- In addition, if the sum of each column is 1, the matrix is called *doubly stochastic*
  - P^T is also a valid transition probability matrix
  - Can jump both $i \to j$ and $j \to i$
  - Not a requirement for Markov chains

Markov Chains 8

- Suppose the initial probability to find X is any of its states is given by vector $a = (a_1, a_2, \ldots)$:

$$P(X_0 = i) = a_i$$

- Further denote by $a^{(n)} = ({a_1}^{(n)}, {a_2}^{(n)}, \ldots)$ the vector of probabilities to find X_n in each state i at time n:

$$P(X_n = i) = a_i^{(n)}$$

- Next, define n-step transition probabilities:

$$p_{ij}^{(n)} = P(X_n = j | X_0 = i)$$

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- Matrix $P^{(n)}$ consists of probabilities $(p_{ij}{}^{(n)})$
- Example: for the coin-toss X_n, compute $p_{i,i+2}^{(3)}$
- Solution:
  - There must be two heads in 3 tosses and one tail:

$$p_{i,i+2}^{(3)} = (1-p)p^2 + p(1-p)p + p^2(1-p) = 3p^2(1-p)$$

  - You could have also used the binomial distribution:

$$p_{i,i+2}^{(3)} = \binom{3}{2}(1-p)p^2 = 3p^2(1-p)$$

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- Now, we obtain $P^{(n)}$ in closed form
- Write: $P(B) = \sum_i P(BA_i), \{A_i\}$ is a partition

$$\begin{aligned} p_{ij}^{(n+s)} &= P(X_{n+s} = j | X_0 = i) \\ &= \sum_k P(X_{n+s} = j, X_n = k | X_0 = i) \\ &= \sum_k P(X_{n+s} = j | X_n = k, X_0 = i) P(X_n = k | X_0 = i) \end{aligned}$$

- Using the Markov property:

$$\begin{aligned} p_{ij}^{(n+s)} &= \sum_k P(X_{n+s} = j | X_n = k) P(X_n = k | X_0 = i) \\ &= \sum_k P(X_s = j | X_0 = k) P(X_n = k | X_0 = i) \end{aligned}$$

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- Thus we get the Chapman-Kolmogorov equation:

$$p_{ij}^{(n+s)} = \sum_k p_{ik}^{(n)} p_{kj}^{(s)}$$

  - Notice that this is a product of the i-th row of $P^{(n)}$ and the j-th column of $P^{(s)}$

- This leads to:

$$P^{(n+s)} = P^{(n)} P^{(s)}$$

- Applying this once for $n = s = 1$, we have $P^{(2)} = P^2$ and recursively expanding:

$$P^{(n)} = P^n$$

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- Finding the probability that a chain has moved from state i to any state j involves matrix multiplication
- It is also simple to express the probability that at time n, the chain is in any of its states:

$$a^{(n)} = a^{(n-1)}P$$

  - Sketch of proof:

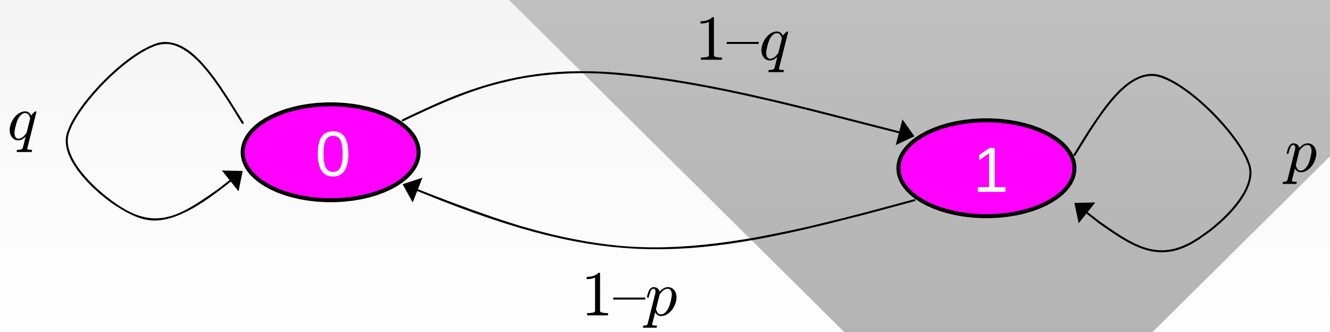
$$a_i^{(n)} = P(X_n = i) = \sum_k P(X_{n-1} = k)p_{ki} = \sum_k a_k^{(n-1)}p_{ki}$$

- We can also write:

$$a^{(n)} = aP^n$$

Markov Chains 13

- Markov chains are often used to model simple processes with a small number of states
- One such example is packet loss
  - Suppose we write 0 when there is no loss (or error) in the channel and 1 when there is
  - We run some traffic over a lossy channel and obtain a packet loss pattern: 0001100000101011110000
  - Design a Markov chain for this loss process



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- There are 22 bits and 21 transitions
  - Counting all possible transitions, we have

$$p_{00} = q = 9/13, \; p_{11} = p = 1/2$$

- Q: assuming that packet loss follows this Markov chain, what is the probability to lose a burst of at least k packets starting from a random time $n \gg 1$?
- A: if we know $P(X_n = 1)$ at time n, then:

$$P(k \; burst) = P(X_n = 1)p^{k-1}$$

- Next, we derive probabilities $P(X_n = i)$ for $n \to \infty$

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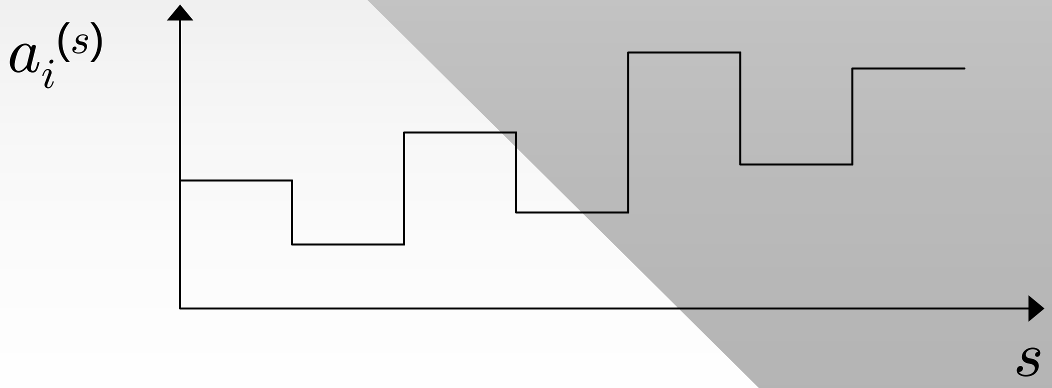
- Recall that $a_i^{(s)}$ is the probability to find the chain in state i after s steps given that it started in a
  - Note that the limit of $a_i^{(s)}$ may not exist as $s \to \infty$
  - Instead, we are interested in the time-average of $a_i^{(s)}$ since it represents the fraction of time the chain spends in state i
- This can be viewed as a time-averaged integral of this curve:



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- More specifically, define the fraction of time the chain spends in state i:

$$\pi_i = \lim_{n\to\infty} \frac{1}{n} \sum_{s=1}^{n} a_i^{(s)}$$

- Existence of this limit is not obvious and will be established next time using renewal theory
  - For now, we assume a stronger condition:

$$\exists \lim_{s\to\infty} a_i^{(s)} = \zeta_i$$

  - Notice that if this limit in fact exists, it equals the one above:

$$\zeta_i = \pi_i$$

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- Using this simpler definition of the probability to find the process in any state i, we have:

$$a^{(s)} = a^{(s-1)}P \Rightarrow \lim_{s\to\infty} a^{(s)} = \lim_{s\to\infty} a^{(s-1)}P$$

- Since both limits are the same, we have

$$\zeta = \zeta P$$

  - where ζ is the left eigen-vector of P

$$\zeta = (\zeta_0, \zeta_1, \ldots)$$

- Values ζ_i are also called *stationary probabilities*
  - Also note that if the chain starts with $a = \zeta$, it follows that $a^{(n)} = \zeta$ for all n

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- Going back to the packet loss example
  - We first construct matrix P

$$P = \begin{pmatrix} q & 1-q \\ 1-p & p \end{pmatrix}$$

  - Next solve the system of equations

$$\zeta = \zeta P = \begin{pmatrix} \zeta_0 q + \zeta_1(1-p) \\ \zeta_0(1-q) + \zeta_1 p \end{pmatrix}$$

  - Rewriting, we get a single equation:

$$\begin{cases} \zeta_0 = \zeta_0 q + \zeta_1(1-p) \\ \zeta_1 = \zeta_0(1-q) + \zeta_1 p \end{cases} \Rightarrow \frac{\zeta_0}{\zeta_1} = \frac{1-p}{1-q}$$

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- Recalling that $\zeta_0 + \zeta_1 = 1$, we have:

$$\begin{cases} \zeta_0 = \frac{1-p}{2-(p+q)} \\ \zeta_1 = 1 - \zeta_0 = \frac{1-q}{2-(p+q)} \end{cases}$$

- Which leads to

$$\begin{cases} \zeta_0 = 0.619 \\ \zeta_1 = 0.381 \end{cases}$$

  - Long-term, 61.9% of all packets are transmitted uncorrupted/without loss, 38.1% are dropped by the network

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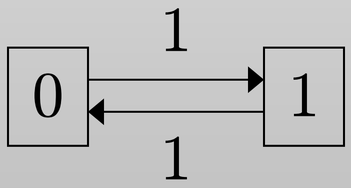
- Let us construct an example when ζ does not exist, but π (time-average) does
  - Assume a deterministically alternating chain with $p_{01} = p_{10} = 1$ and $a_0 = 1 = 1 - a_1$
  - Then, for all even values of s, the chain is always in state 0 and for all odd s, it is always in state 1
  - Thus, the limit of $a^{(s)}$ does not exist



- At the same time, the summation limit is well-defined and correctly gives the time spent in each state:

$$\pi_0 = \lim_{n \to \infty} \frac{1}{n} \sum_{s=1}^{n} a_0^{(s)} = 1/2$$

Wrap-up

- Under the Markov loss chain, what is the probability to receive exactly k packets (no more, no less) starting at some random time n?

$$P(k \ are \ good) = \pi_0 q^{k-1}(1-q)$$

  - This is a geometric-like distribution

- Notice that burst lengths of both 0s and 1s have exponential tails
  - More generic models allows substantially longer (i.e., Pareto) bursts using non-Markovian dynamics