CSCE 619-600
Networks and Distributed Processing
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Renewal Process Theory

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Agenda

- Stochastic processes
  - Definitions
  - Homework #2 examples
- Renewal processes
- More examples

Process Theory

- In many cases, we deal with more than just a single random variable
  - This arises in repeated experiments such as coin tosses, or recurring phenomena such as bus arrivals
- Definition: a *process* is an infinite collection of random variables
  - There are two types of processes – discrete and continuous
- A discrete process $\{X(n)\}$ is a *countable* sequence of random variables, where $X(n)$ is the n-th variable (often written as X_n)
- If the number of variables is *uncountable*, we have a continuous process $\{X(t)\}$ or $\{X_t\}$

Process Theory 2

- The simplest example are Bernoulli coin tosses
  - Sequence X_1, X_2, …is a process
  - Since these are iid (independent identically distributed) random variables, the mean of X_n does not depend on n
  - We can thus write $E[X_n] = p$
- It is possible to construct a dependent process
  - Variable X_n depends on previous values X_1, …, X_{n-1}
  - For example, autoregressive process $X_n = X_{n-1} + v(n)$, where $v(n)$ are some random variables (often called noise)
- Instance of the n-th bus arrival is also a process:

$$Z_n = \sum_{i=1}^{n} X_i = Z_{n-1} + X_n$$

Process Theory 3

- Another example is the evolution of wealth
  - Suppose the more money you have during year $n-1$, the more its increase in year n
  - Write:

$$X(n) = X(n-1) + w(n)X(n-1)$$

  - where $w(n)$ is some multiplicative random noise (suppose it is uniform in $[0,1]$)
- Then what is the distribution of your wealth in year n?
  - Suppose $10{,}000$ people start with $X(0) = \$1$; how much skew will there be in year n?
  - Can this model capture the wealth distribution of modern societies?

Process Theory 4

- Clearly, the growth of $X(n)$ is exponential in some "average" sense
  - Its mean is computed easily:

$$E[X(n)] = E[X(0)] \prod_{i=1}^{n} (1 + E[w(i)])$$

- Assuming that $w(i)$ are iid, we have:

$$E[X(n)] = E[X(0)](1 + E[w(1)])^n$$

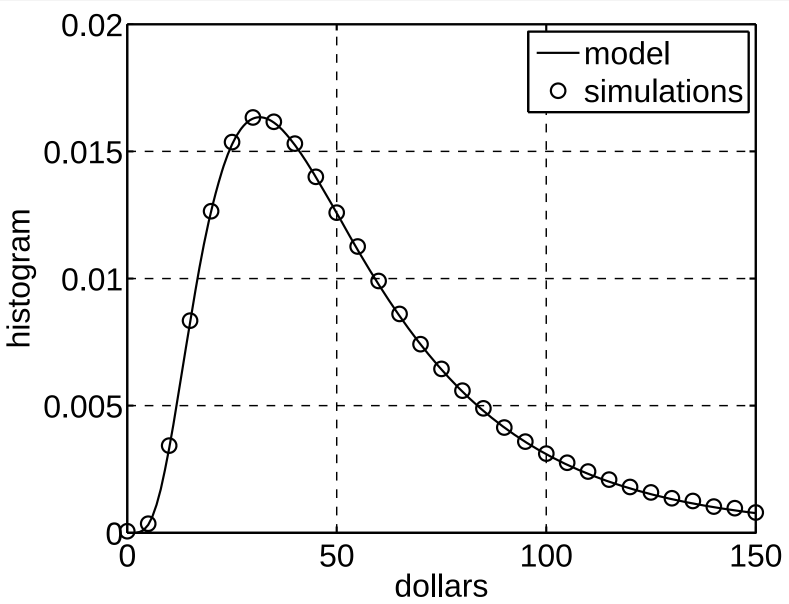
- Average wealth increases multiplicatively from $X(0)$
  - The longer one lives, the more money they are expected to have in this model

Process Theory 5

- What can be said about the distribution of $X(n)$?
  - Problem addressed in hw #2
- Your goal is to derive the distribution of $X(n)$ and confirm in simulation that your results are accurate
- You need to use lognormal random variables
  - Variable X is said to be lognormal if there exists some Gaussian random variable Y such that $X = e^Y$
  - Excel has its CDF $F(x) = \texttt{lognormdist}(x, E[Y], \sigma_Y)$
- In this case, you can only match the histogram since the CDF function does not exist in closed-form
  - A histogram counts the number of samples generated by a random variable in each bin of certain size

Process Theory 6

- How to compute histogram of the model to compare it to that obtained in simulations?
  - Assume bin $[a_i,\ a_{i+1})$
  - Then the model predicts that the fraction of samples in this bin will be $F(a_{i+1}) - F(a_i)$
- Alternative methods exist for directly matching the PDF of a distribution, but they are more complex

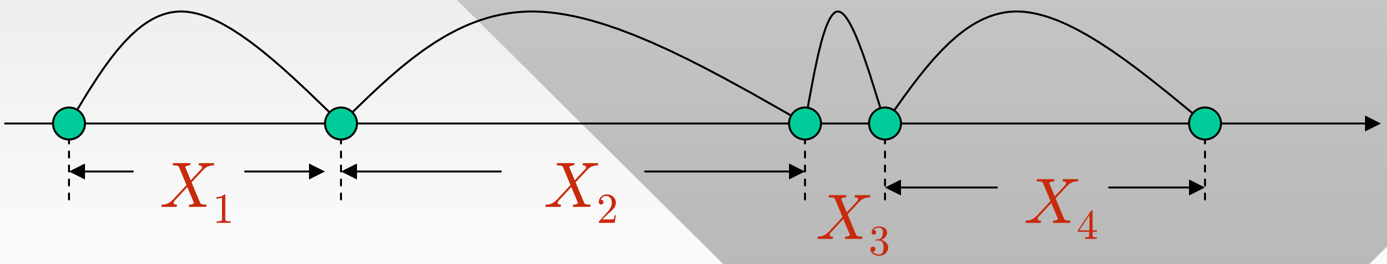


Renewal Processes

- One important area of process theory deals with renewals (periodic events)
  - Analysis of queues in routers, user arrivals in P2P systems, request backlog in webservers, etc.
- Suppose our system exhibits *recurring* behavior
  - Recurring means that the same situation repeats over time at random time instances
  - Bus arrival is one example from homework #1
- Analysis focuses on queue size distributions, overflow events, and various wait times
  - One common model is an observer that examines the system at some point $t \to \infty$ (asymptotic behavior)

Renewal Processes 2

- Each occurrence of an event is called a *renewal*
  - Delays between adjacent renewals are given by a sequence of iid random variables X_1, X_2, …
  - This forms a discrete process X_n
- Here is an illustration:



Renewal Processes 3

- The time (epoch) of the n-th renewal (bus arrival):

$$Z_n = \sum_{j=1}^{n} X_j$$

- Usually $E[X_j]$ is finite and there exists a renewal rate

$$\mu = \frac{1}{E[X_j]}$$

- It is generally convenient to assume that X_1 has a different distribution from X_n, $n > 1$
  - Allows construction of stationary processes

Renewal Processes 4

- Let X_1 have a CDF $A(x)$ (density $a(x)$) and the remaining X_j have a CDF $F(x)$ (density $f(x)$)
- Then, the distribution of time before the second renewal:

$$P(Z_2 \le t) = P(X_1 + X_2 < t) = \int_0^t A(t-u) f(u) du$$

- Differentiating the above expression, the density of Z_2 is given by:

$$[P(Z_2 \le t)]' = \int_0^t a(t-u) f(u) du$$

  - which is standard covolution

Renewal Processes 5

- To expand further, denote by $F^{(2)}(t) = F * F$ and by $F^{(n)}(t) = F^{(n-1)}(t) * F(t)$ an n-fold convolution
  - This represents the distribution of the sum of n iid variables, each with CDF $F(x)$
- Thus, since $Z_n = X_1 + X_2 + \ldots + X_n$, we get:

$$P(Z_n \leq t) = A(t) * F^{(n-1)}(t)$$

  - which is the distribution of the n-th renewal time
- Not a simple metric to compute!
  - But it gets worse

Renewal Processes 6

- Define *renewal process* $M(t)$ to be the number of renewals in the interval $[0, t]$:

$$M(t) = \max\{n \geq 0 : Z_n \leq t\}$$

- $M(t)$ is a continuous-time, discrete-state process
- We can express its tail distribution using Z_n:

$$P(M(t) \geq n) = P(Z_n \leq t) = A(t) * F^{(n-1)}(t)$$

- This is the same n-fold convolution

Renewal Processes 7

- Define renewal function $m(t) = E[M(t)]$ to be the expected number of renewals in $[0, t]$
- Recalling the expectation of a non-negative random variable as the sum of the tail

$$E[X] = \sum_{j=0}^{\infty} P(X > j)$$

  - we get

$$m(t) = E[M(t)] = \sum_{j=1}^{\infty} A(t) * F^{(j-1)}(t)$$

- This is not very helpful at all (notice the infinite convolution)

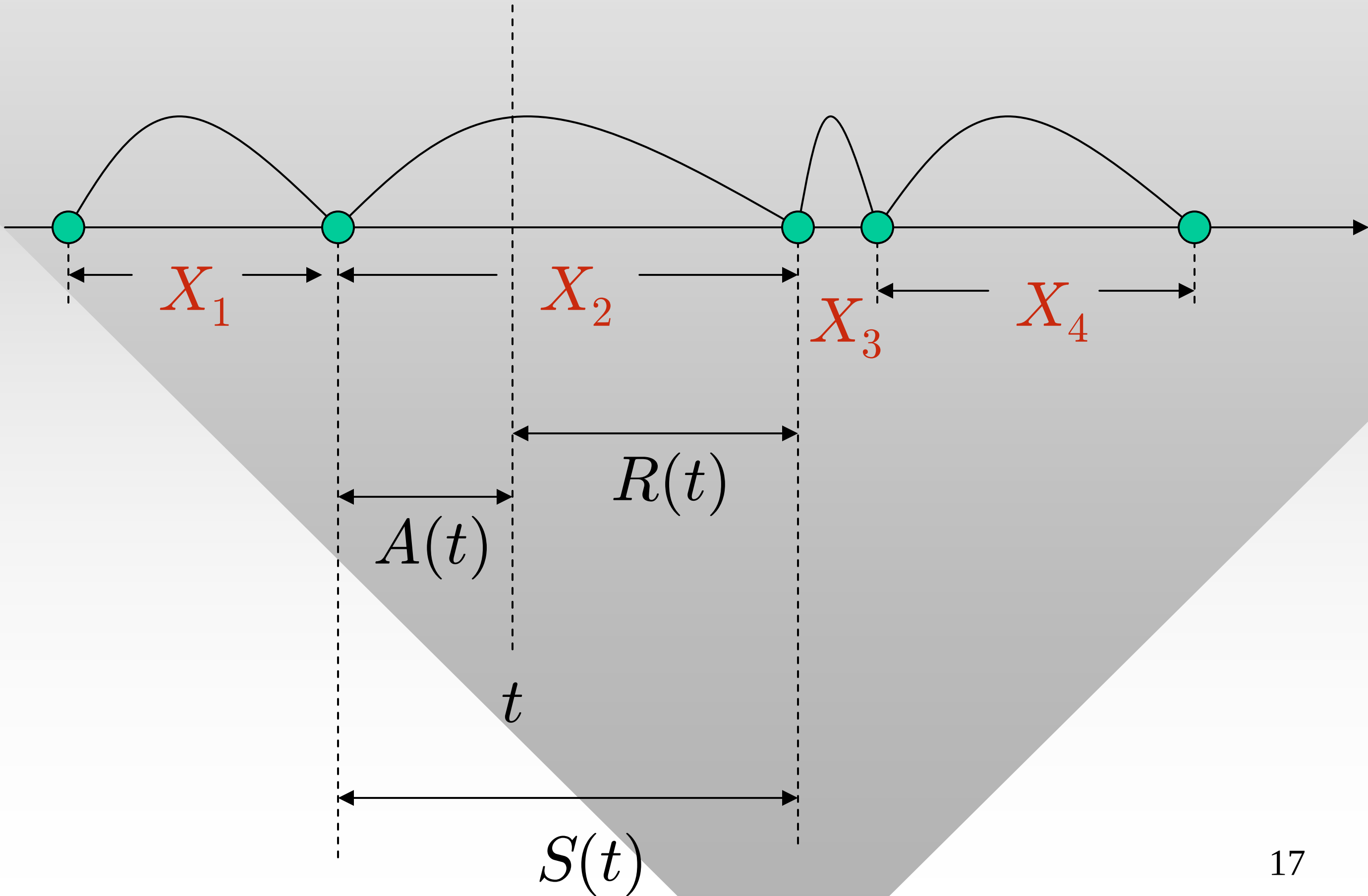
Renewal Processes 8

- In fact, if we know little about $F(x)$, we cannot compute even the expected number of arrivals in $[0, t]$
- For this reason, much of renewal theory deals with asymptotic results
  - This means that we are interested in systems that have evolved sufficiently long and points t much larger than 0, in which case we have the following result
- The Elementary Renewal Theorem:

$$\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{t\to\infty} \frac{m(t)}{t} = \mu$$

Renewal Processes 9

- Other metrics of interest
  - The age $A(t)$ of the process at time t and the residual life $R(t)$ until the next renewal



Renewal Processes 10

- The age is the delay since the last bus left and the residual life is the wait time until the next bus
- We can express both using $M(t)$:

$$R(t) = Z_{M(t)+1} - t \qquad A(t) = t - Z_{M(t)}$$

- We can also define spread $S(t)$ to be the current delay between the buses sampled randomly at time t

$$S(t) = A(t) + R(t) = Z_{M(t)+1} - Z_{M(t)}$$