CSCE 619-600 Networks and Distributed Processing Spring 2017

Renewal Process Theory II

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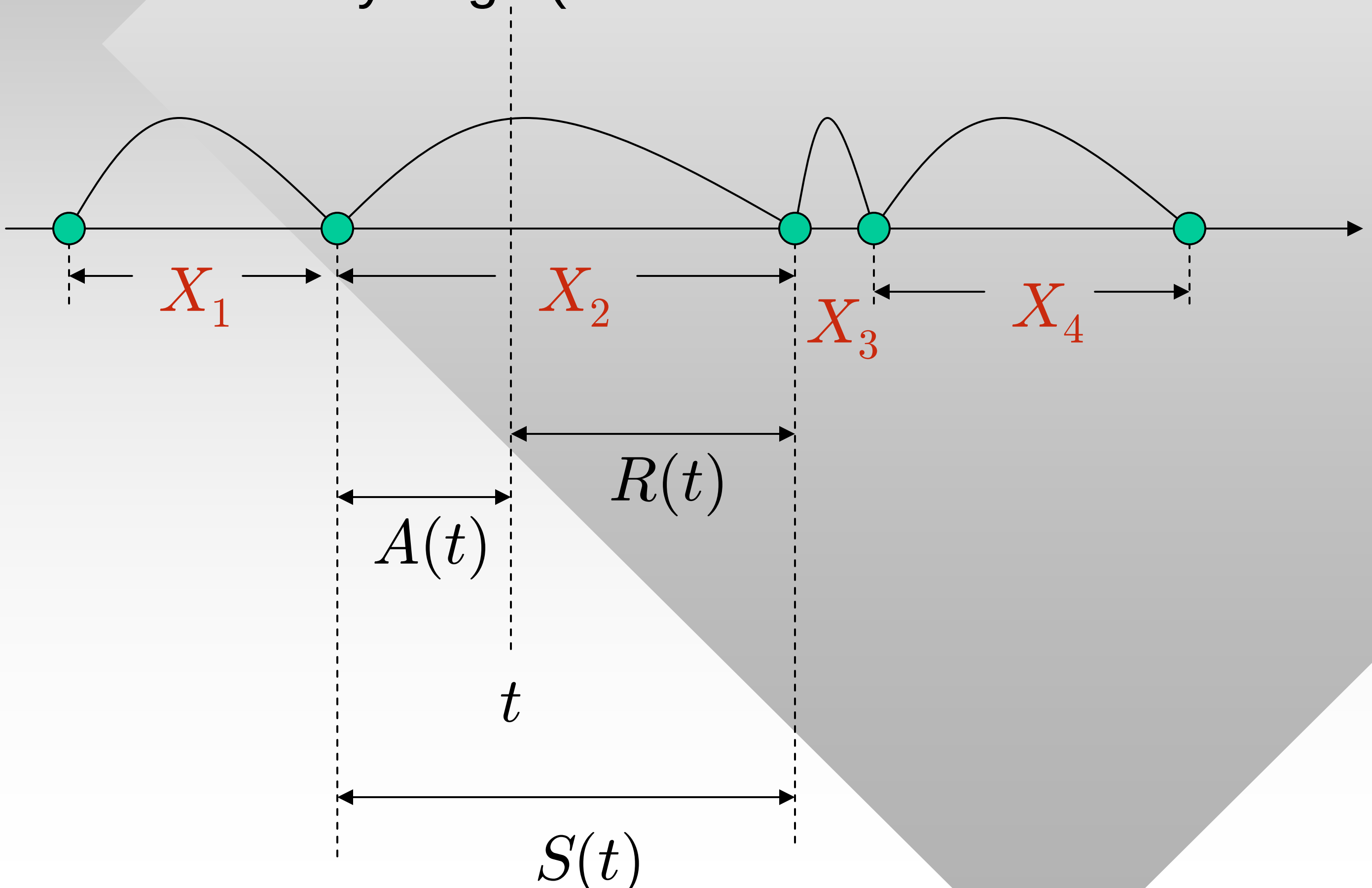
Agenda

- Inspection paradox
- Renewal-reward processes
  - Renewal-reward theorem
  - Examples
- Distribution of residuals and age
  - Derivation and examples
- Distribution of spread
  - Examples

Inspection Paradox

- Recall from the previous lecture
  - Age $A(t)$, residual delay $R(t)$, and spread $S(t)$
  - Time t is very large (think of it as uniform between 0 and ∞)



Inspection Paradox 2

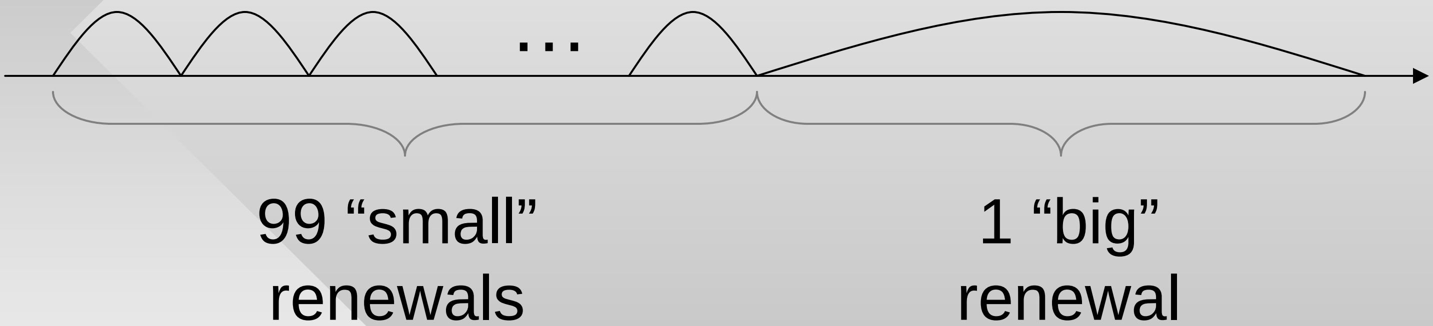
- The "inspection paradox"
  - When we randomly examine a renewal process at time t, the average spread $S(t)$ can be much larger than $E[X_j]$
- This can be shown through an example
  - Consider two types of inter-bus delays:

$$P(X_j = 1) = 0.99, P(X_j = 100) = 0.01$$

- This means that 99% of all renewal intervals are 1 minute and the remaining 1% are 100 minutes
  - What is the probability to observe interval $X_j = 100$ at some large time instance t?
- Clearly, not 1%

Inspection Paradox 3

- Notice that in the “average sense,” the process has this structure:



- Thus, each 199 minutes (on average) contain one large and 99 small intervals
  - Therefore, the probability that t randomly “lands” into a large interval is 100/199 $\approx$ 50% instead of 1%

Inspection Paradox 4

- A random observer is more likely to inspect the process during large intervals and experience longer wait times than if he/she arrived at renewal points Z_n
- Definition (pg. 22)
  - X is *stochastically larger* than Y if

$$P(X > t) \geq P(Y > t), \forall t$$

  - For example, uniform X in $[3,5]$ is stochastically larger than uniform Y in $[2,4]$
- Theorem: random variable $S(t)$ is stochastically larger than X_j

Inspection Paradox 5

- We now have an intuitive explanation of why Pareto wait times in the bus example became larger
- This is a consequence of very large samples X_j in the Pareto distribution during which we are more likely to inspect the system
- Example:
  - Compute the expected wait time for the example shown two slides back
  - Recall:

$$E[X] = \sum_y E[X|Y = y]P(Y = y)$$

Inspection Paradox 6

- Thus:

$$E[W] = E[W|small]P(small) + E[W|large]P(large)$$

  - where

$$P(small) = 99/199, P(large) = 100/199$$

- Putting the pieces together:

$$E[W] = \frac{1}{2} \times \frac{99}{199} + \frac{100}{2} \times \frac{100}{199} = 25.37...$$

- Also note that $E[X_j]$ is only 1.99 minutes
  - This means that buses arrive at the rate of 30 per hour; however, your expected wait is over 25 minutes!

Renewal Rewards

- Our goal today is to derive the distribution of residual waiting time $R(t)$ and spread $S(t)$
  - Before we do that, we need several results from renewal rewards, which is a segment of renewal process theory
- Suppose each renewal j earns reward P_j
  - Sequence {P_1, P_2, …} consists of iid random variables
- Then, the cumulative reward in $[0, t]$ is

$$C(t) = \sum_{j=1}^{M(t)} P_j$$

- Technical note
  - Rewards P_j may depend on X_j, but not X_i, $i \neq j$

Renewal Rewards 2

- Let the *expected reward* be $c(t) = E[C(t)]$
- The Renewal-Reward Theorem:

$$\lim_{t\to\infty} \frac{C(t)}{t} = \lim_{t\to\infty} \frac{c(t)}{t} = \frac{E[P_j]}{E[X_j]} = \mu E[P_j]$$

- Example:
  - You are taking a long test that lasts t time units
  - Suppose it takes X_j seconds to solve problem j
  - You earn P_j points for solving the j-th problem
  - What is your score at the end of the test?
  - Approximately

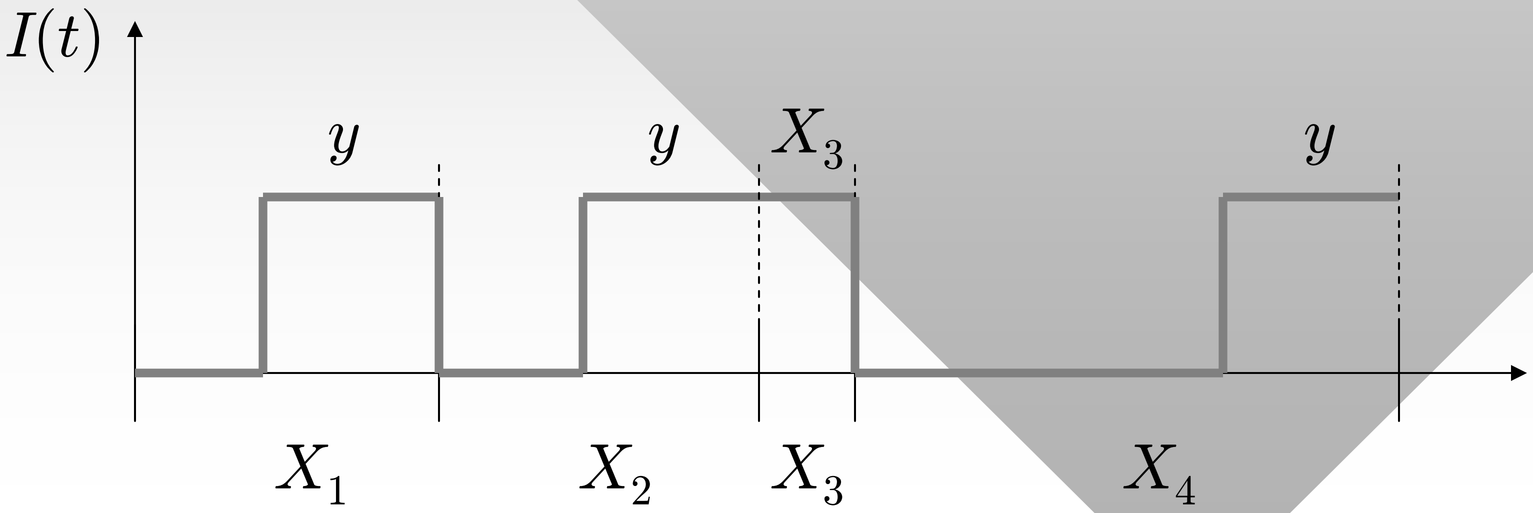
$$t\frac{E[P_j]}{E[X_j]}$$

Distribution of Residuals

- We are ready to derive the distribution of $R(t)$
- Fix $y \geq 0$ and define indicator process:

$$I(t) = \begin{cases} 1 & R(t) \leq y \\ 0 & otherwise \end{cases}$$

- Graphically, it looks like this:



Distribution of Residuals 2

- If we throw a random point t on the interval $[0, T]$, what is the probability to hit a segment with $I(t) = 1$?

$$P(R(t) \le y) = \frac{1}{T}\int_0^T I(u)du$$

- Setting R as the limit of $R(t)$ for $t \to \infty$

$$P(R \le y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T I(u)du$$

- All we now have to do is compute this limit
  - Define reward P_j for renewal j to be:

$$P_j = \int_{Z_{j-1}}^{Z_j} I(u)du = \begin{cases} X_j & X_j \le y \\ y & X_j > y \end{cases}$$

  - Alternatively, $P_j = \min(X_j, y)$

Distribution of Residuals 3

- Next observe that the integral of $I(u)$ is actually the cumulative reward at time T:

$$C(T) = \sum_{j=1}^{M(T)} P_j = \int_0^T I(u)du$$

- We neglect special cases when T does not fall on the boundary of Z_j (i.e, partial rewards) as their contribution tends to zero for large T
- Next, apply the renewal reward theorem to $C(T)$:

$$\lim_{T\to\infty} \frac{C(T)}{T} = \lim_{T\to\infty} \frac{1}{T}\int_0^T I(u)du = \frac{E[P_j]}{E[X_j]}$$

Distribution of Residuals 4

- Observe that the middle term in the last equation is $P(R \leq y)$ and re-write:

$$P(R \leq y) = \frac{E[P_j]}{E[X_j]}$$

  - What is left is to obtain $E[P_j]$
- Lemma: Suppose X is a non-negative variable and $a > 0$ is some constant. Then

$$\begin{aligned} E[\min(X,a)] &= \int_0^\infty P(\min(X,a) > x)dx \\ &= \int_0^a P(\min(X,a) > x)dx \\ &= \int_0^a P(X > x)dx \end{aligned}$$

  - Reason: for any $x \leq a$, it follows that $X > x$ iff $\min(X,a) > x$

Distribution of Residuals 5

- Combining the pieces:

$$P(R \leq y) = \frac{1}{E[X_j]} \int_0^y P(X_j > x) dx$$

- Or in a more digestible form:

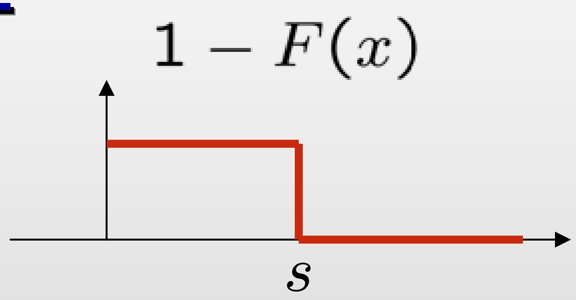
$$P(R \leq y) = \frac{1}{E[X_j]} \int_0^y (1 - F(x)) dx$$

- Note: the same technique applies to age:

$$P(A \leq y) = \frac{1}{E[X_j]} \int_0^y (1 - F(x)) dx$$

Distribution of Residuals 6

- Examples:
  - Constant inter-bus delay s:



$$P(R \leq y) = \begin{cases} y/s & y \leq s \\ 1 & y > s \end{cases}$$

uniform

  - Exponential:

$$P(R \leq y) = \lambda \int_0^y e^{-\lambda x} dx = 1 - e^{-\lambda y}$$

exponential

  - Pareto:

$$P(R \leq y) = \frac{\alpha - 1}{\beta} \int_0^y (1 + x/\beta)^{-\alpha} dx = 1 - (1 + y/\beta)^{1-\alpha}$$

Pareto

Distribution of Residuals 7

- This explains the observations in homework #1
  - Inter-bus delays with Pareto $\alpha = 3$ had wait times that were Pareto with $\alpha = 2$
  - It also confirms that when the delay between buses is constant, your expected wait is uniform in $[0, s]$
- The final important result to derive is the expected wait time $E[R]$
- Define the residual CDF (also called the *equilibrium* distribution) of F:

$$F_R(y) := P(R \leq y) = \frac{1}{E[X_j]} \int_0^y (1 - F(x))dx$$

Distribution of Residuals 8

- Then the density of wait time is:

$$f_R(y) = F_R'(y) = \frac{1 - F(y)}{E[X_j]}$$

- Next, we compute the expected wait time:

$$E[R] = \int_0^\infty y f_R(y) dy = \frac{1}{E[X_j]} \int_0^\infty y(1 - F(y)) dy$$

- To solve the integral, notice:

$$E[X^2] = \int_0^\infty P(X^2 > x) dx = \int_0^\infty P(X > x^{1/2}) dx$$

Distribution of Residuals 9

- Substituting $u = x^{1/2}$ ($dx = 2udu$):

$$E[X^2] = \int_0^\infty 2uP(X > u)du = 2\int_0^\infty u(1-F(u))du$$

- Recalling the equation on $E[R]$:

$$E[R] = \frac{1}{E[X_j]}\int_0^\infty y(1-F(y))dy$$

  - We have:

$$E[R] = \frac{E[X_j^2]}{2E[X_j]}$$

Distribution of Residuals 10

- Examples:
  - Constant inter-bus delay:

$$E[R] = \frac{s^2}{2s} = \frac{s}{2}$$

  - Exponential:

$$E[X^2] = Var[X] + E^2[X] = 2/\lambda^2 \Rightarrow$$

$$E[R] = \frac{2/\lambda^2}{2/\lambda} = 1/\lambda = E[X_j]$$

  - Pareto: derive at home

Distribution of Spread

- Using similar techniques, we can derive the limiting distribution of spread $S(t)$
- The only difference is that we define

$$P_j = \begin{cases} X_j & X_j \leq y \\ 0 & X_j > y \end{cases}$$

- The rest of the derivations are similar and lead to:

$$F_S(y) := P(S \leq y) = \frac{1}{E[X_j]}\left(yF(y) - \int_0^y F(x)dx\right)$$

  - and its density is:

$$f_S(y) = \frac{yf(y)}{E[X_j]}$$

Wrap-up

- Example
  - Spread distribution for exponential X_j:

$$f_S(y) = \frac{yf(y)}{E[X_j]} = \frac{y\lambda e^{-\lambda y}}{1/\lambda} = \lambda^2 y e^{-\lambda y}$$

- Interestingly, this is the same density as that of a sum of two exponential random variables, i.e., Erlang(2)
  - Why was this intuitively expected?
- For exponential X_j, average spread $E[S] = 2E[X_j]$
  - In general,

$$E[S] = E[R] + E[A] = 2E[R] = \frac{E[X_j^2]}{E[X_j]}$$