# CSCE 619-600
# Networks and Distributed Processing
# Spring 2017

## Random Graphs III

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March 28, 2017

# Agenda

- Random-graph models
  - $G(n, k_{\text{out}})$
  - $G(n, r)$
- Percolation of monotone properties
- Relationship between connectivity and degree

# Random Graph Models 9

- The next model is $G(n, k_{\text{out}})$
  - Each node randomly selects $k$ nodes from the remaining $n-1$ vertices to be its neighbors
  - This is a *directed* graph
  - The out-degree is $k$, but what is the in-degree?
- The probability that node $v$ is selected by $u$ is:

$$p_{uv} = 1 - \prod_{i=1}^{k} \left(1 - \frac{1}{n-i}\right) = \frac{k}{n-1}$$

- Since there are $n-1$ nodes $u$, the in-degree of $v$ is:

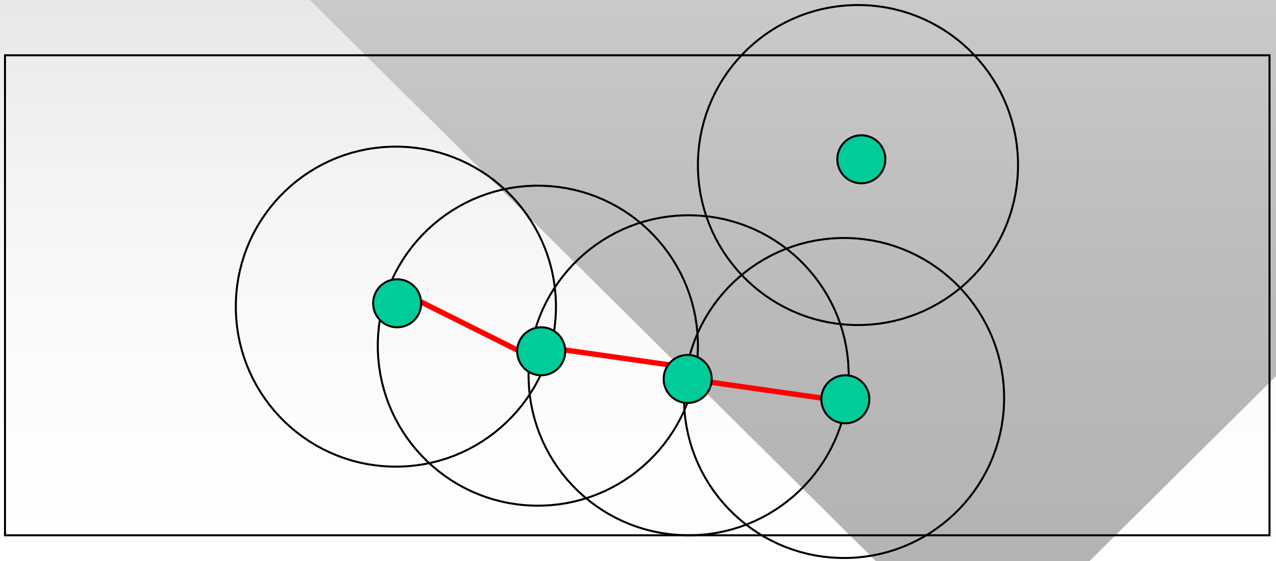
$$d_v^{in} \sim B(n-1, p_{uv})$$

# Random Graph Models 10

- Easy to notice that $E[d_v^{in}] = k$
- Graphs $G(n,k_{\text{out}})$ are often found in user-driven networks with freedom to choose neighbors
  - A prime example would be a P2P system, where each joining user randomly selects $k$ other peers to be his/her neighbors
- Another version of this model is the social network
  - Each person randomly selects $k$ other people in the world to be their friends
- Q: What is the probability that each person in an $n$-node network is connected to every other one through a chain of acquaintances?

# Random Graph Models 7

- The final model is $G(n,r)$
  - A fairly recent development that models wireless ad-hoc networks (i.e., users form a graph using each other's computers as routers)
  - Two nodes $u$ and $v$ are linked if and only if $||u - v|| \leq r$
  - Parameter $r$ is the communication radius and || . || is some norm (usually Euclidean) in the 2D space



# Random Graph Models 7

- Formalization
  - Coordinates $(x,y)$ of each node in $G(n,r)$ are iid uniform random variables in $[0,A]$
  - Nodes scattered in a 2D square whose area is $A^2$
  - Edge $(u,v)$ exists iff distance between $u$ and $v$ is less than $r$
- Define *node density* $\rho$:

$$\rho = \frac{n}{A^2}$$

- Number of nodes in an area of size $S$?
  - Binomial with parameters $(n,\ S/A^2)$
  - The mean is simply $\rho S$

# Random Graph Models 8

- Q: What is the expected number of neighbors (degree) of each node?

$$E[d_i] = \rho \pi r^2 - 1$$

- Q: How many wireless sensors with a communication radius of 200 feet does one need to scatter in a zone 50x50 miles to obtain a connected network with high probability?

- This relatively simple question requires rather complex mathematical tools, which we omit here
  - Derivations can be found in Bollobas "Random Graphs," *Cambridge University Press*, 2001 and Penrose "Random Geometric Graphs," *Oxford University Press*, 2003

# Graph Properties

- Definition: a property $Q$ is *monotone* if whenever $G$ has $Q$, every graph with the same number of nodes containing $G$ also has $Q$
  - Monotonicity applies to addition of new edges
- Which of these are monotone?
  - Graph contains a triangle (3-cycle)
  - Graph contains a complete sub-graph of order 5
  - The number of edges is odd
  - Graph is connected
- We will understand connectivity of random graphs using an edge-growth process similar to $G(n,M)$

# Graph Properties 2

- Theorem: if $Q$ is monotone and $M_1 < M_2$, then:

$$P(G(n, M_1) \text{ has } Q) \leq P(G(n, M_2) \text{ has } Q)$$

- Theorem: if $Q$ is monotone and $p_1 < p_2$, then

$$P(G(n, p_1) \text{ has } Q) \leq P(G(n, p_2) \text{ has } Q)$$

- Definition: we say that almost every (a.e.) graph has $Q$, if $P(\text{graph has } Q) \to 1$ as $n \to \infty$
  - We may occasionally produce graphs in which $Q$ does not hold, but the fraction of such graphs tends to $0$ as $n \to \infty$
- Alternatively we may say that $Q$ holds almost surely
  - Meaning with probability $1 - o(1)$ as $n \to \infty$

# Graph Properties 3

- Erdos was first to discover that certain monotone properties in $G(n, M(n))$ appeared "suddenly"
  - There is a threshold function $M^*(n)$ such that:

$$\frac{M(n)}{M^*(n)} \to 0 \Rightarrow P(G(n, M(n)) \text{ has } Q) \to 0$$

  - and:

$$\frac{M(n)}{M^*(n)} \to \infty \Rightarrow P(G(n, M(n)) \text{ has } Q) \to 1$$

- <u>Example</u>: $M^*(n) = n^{1.5}$; find some examples of $M(n)$ for each of the two cases above

# Graph Properties 4

- Threshold functions are not unique and differ by a fixed factor:
  - For any two threshold functions

$$M_1^*(n) = O(M_2^*(n)), \;\; M_2^*(n) = O(M_1^*(n))$$

- To better understand behavior of random graphs, examine their evolution as defined below:
  - Graph $G_0$ starts with $n$ nodes and no edges
  - $G_t$ is obtained from $G_{t-1}$ by adding a random edge
  - $G_t$ has $t$ edges and is called a *graph process*
- Clearly, $G_0 \subset G_1 \subset \ldots \subset G_t$

# Graph Properties 5

- One particularly interesting monotone property $Q$ of a graph process $G_t$ is its *connectivity*
  - Once the graph is connected at some time $t_Q$, it stays connected throughout the remainder of the process
- More formally, $t_Q$ is called the *hitting time* of $Q$:

$$t_Q = \min\{t \geq 0 : G_t \text{ has } Q\}$$

- Theorem: the hitting time is almost always “close” to the threshold function $M^*(n)$ of $Q$
  - The closeness is in the asymptotic sense

# Graph Properties 6

- Our first main result is the threshold function for graphs $G(n,p)$ and $G(n,M)$
  - Both thresholds are equivalent to accumulation of average degree $\log(n)$
  - However, we can do even better
- Theorem: assume that

$$M = \frac{n(\log n + c(n))}{2}, \; p = \frac{\log n + c(n)}{n}$$

  - then:

$$P(G \text{ is connected}) \to e^{-e^{-c(n)}}$$

# Graph Properties 7

$$M = \frac{n(\log n + c(n))}{2}, \quad p = \frac{\log n + c(n)}{n}$$

$$P(G \text{ is connected}) \to e^{-e^{-c(n)}}$$

- As soon as $G_t$ accumulates $n(\log n + c(n))/2$ edges, it becomes connected:
  - With probability $1$ if $c(n) \to \infty$
  - With probability $0$ if $c(n) \to -\infty$
- For example, $M = n(\log n + \text{loglog } n)/2$ is sufficient for connectivity of a.e. graph
  - Similarly, $M = n(\log n - \text{loglog } n)/2$ guarantees that almost no graph is connected
- Q: What is the asymptotic probability of connectivity of random graphs under these conditions?
  - Suppose $p = 2\log(n)/n$, or $p = \log(n)/n$ or $p = \text{sqrt}(n)/n$
  - What if $p = (\log^2(n) - 3\log(n))/n$?

# Graph Properties 8

- It is often convenient to express thresholds in terms of degree
  - Recall that expected degree $E[d] = 2M/n \approx np$
- Restating the same thresholds, we have:

$$E[d] = \log n + c(n) \Rightarrow P(G \text{ is connected}) = e^{-e^{-c}}$$

  - If $c$ tends to $+$ or $-\infty$, we have two options: 1) a.e. graph is connected or 2) a.e. graph is disconnected
- This effect is known as percolation
  - Originated in physics under percolation theory
  - Used to model forest fires, social networks, and various physical/chemical problems

# Graph Properties 9

- In general, how does $G_t$ look for different $t$?
  - This is a stochastic process with a vast number of properties
- Theorem: as time $t$ evolves, $G_t$ goes through the following three stages:
  - 1) If $E[d(t)] < 1$, then a.e. component is a tree (i.e., no cycles in the graph) and the size of each component is $O(\log n)$
  - 2) As soon as $E[d(t)]$ becomes larger than 1 (i.e., $E[d] = c > 1)$ cycles start to emerge and we have the formation of one *giant* component of size $\Theta(n^{2/3})$; all smaller components are of size $O(\log n)$
  - 3) The giant component grows 4 times faster than $t$ and "swallows" all other components by the time $E[d(t)]$ reaches and slightly exceeds $\log n$