CSCE 619-600
Networks and Distributed Processing
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Random Graphs IV

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Agenda

- Connectivity (continued)
- Diameter of random graphs

Graph Properties 10

- Interestingly, we can also deduce connectivity by observing the smallest degree in the graph
  - Define $\delta(G)$ to be the smallest degree in G
- Theorem: the probability of obtaining a connected random graph is equal to the probability that there are no isolated vertices:

$$P(G \text{ is connected}) = P(\delta(G) \geq 1) = e^{-e^{-c}}$$

- Important to remember that this only applies to families of random (not deterministic) graphs

Graph Properties 11

- Vertex connectivity $m(G)$
  - Definition: a graph is m-connected, i.e., $m(G) = m$, if the removal of any $m-1$ nodes keeps the graph connected and no larger m with this property exists
  - Useful concept in reliability and fault-tolerance analysis
- For Erdos graphs, $P(G \text{ is } m\text{-node-connected})$ equals $P(\delta(G) = m)$, but a stronger result is possible
- Theorem: for integer $k > 0$ and $E[d] = \log n + k \log\log n + x$, the connectivity of the graph is

$$P(m(G) = k) \to 1 - e^{-e^{-x}/k!},$$

$$P(m(G) = k+1) \to e^{-e^{-x}/k!}$$

Graph Properties 12

$$P(m(G) = k) \to 1 - e^{-e^{-x/k!}},$$
$$P(m(G) = k + 1) \to e^{-e^{-x/k!}}$$

- Setting $k = 0$, we get the earlier result on connectivity
- Example: design a random $G(n,p)$ graph that tolerates the removal of any 9 vertices and stays connected with probability 1 as $n \to \infty$
  - The goal here is to obtain a 10-connected graph
- The simplest approach is to set:

$$p = \frac{\log n + 10 \log \log n}{n}$$

  - This gives us $1 – 1/e = 63\%$ of 10-connected and 37% of 11-connected graphs
- However, is there a smaller p we can use?

Graph Properties 13

- Since we only need $m(G) = 10$, one option is:

$$p = \frac{\log n + 10 \log\log n - \log\log\log n}{n}$$

  - This guarantees that a.e. graph is 10-connected

- We can do even better using for example:

$$p = \frac{\log n + 9 \log\log n + \log\log\log n}{n}$$

  - Or any other slowly growing function $x(n)\to\infty$

Graph Properties 14

- Random geometric graphs are very similar
- Penrose and Gupta/Kumar proved in the late 1990s that $G(n,r)$ becomes connected as soon as each node obtains $\log n$ neighbors on average
- Theorem: assuming the communication radius r is

$$r = \sqrt{\frac{\log n + c(n)}{\pi \rho}}$$

the $G(n,r)$ graph is connected with probability:

$$P(connected) = e^{-e^{-c(n)}}$$

Graph Properties 15

- Notice that since the average degree is $\rho\pi r^2$, we have:

$$E[d] = \rho\pi r^2 = \rho\pi\frac{\log n + c(n)}{\pi\rho} = \log n + c(n)$$

- Question: for a fixed r, how many nodes are needed in a $1\text{x}1$ square to achieve 87% and 99% connectivity?
  - Represent
  - where

$$E[d] = \log(n) + n\pi r^2 - \log(n)$$

$$c(n) = n\pi r^2 - \log(n)$$

$$P(connected) = e^{-e^{-c(n)}}$$

- Assume $r = 0.1$
  - For 87% connectivity, need 237 nodes
  - For 99%, need 331 nodes

Graph Properties 16

- Finally, $G(n,k_{\text{out}})$ is almost surely (a.s.) k-node-connected as long as $k \geq 2$
  - This is a rather strong result pointing towards high resilience and good connectivity of P2P networks
- Furthermore, k-node-connectivity means that there are at least k node-disjoint paths between any pair of nodes
  - This provides plenty of backup routing options when the main path fails for whatever reason
- In fact, a k-regular graph cannot have node connectivity higher than k

Graph Properties 17

- We finish with the diameter of the graph
- Theorem: assume that the average degree of $G(n,p)$ or $G(n,M)$ is $E[d] = c\log n$; then we have:
  - 1) If $c > 1$, the graph is a.s. connected and its diameter concentrates on several values around:

$$D \approx \frac{\log n}{\log E[d]} = \log_{E[d]} n$$

  - 2) For $c < 1$, the graph is a.s. disconnected and the diameter of its giant component is $D \approx (1+o(1))\log_{E[d]} n$
- Theorem: the diameter of $G(n,\ k_{\text{out}})$ is asymptotically:

$$\log_{E[d]} n(1+o(1)) \leq D \leq \log_{E[d]} n(1+o(1))+1$$

Graph Properties 18

- Optimality of these diameters is not difficult to see
  - A k-regular graph cannot have diameter smaller than $\log_k n$
- Q: Suppose $p = 3\log(n)/n$ in a $G(n,p)$ graph; then each edge is independently failed with probability q
  - What is the probability that the resulting graph is connected?
  - What if each node is failed with probability q?
  - For these two cases, what is the largest q that still allows the resulting graph to stay connected?
- Despite complexity of derivations, random graphs above are still too simple to model real-life networks
  - Next time, we will talk about more realistic models