CSCE 619-600
Networks and Distributed Processing
Spring 2017

Congestion Control II

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April 13, 2017

Agenda

- Goals of congestion control
- Fairness index
- System stability
  - Stationarity
  - Discrete maps and fixed points
- Lyapunov stability
  - Global and local
- Examples
  - Monotonic convergence, regions of stability

Congestion Control

- What are the main problems in designing congestion control?
  - First, we need to understand what congestion control is and what its goal are
- Consider a link of physical capacity C bits per second
- Assume N flows share this link
  - Each flow's sending rate is $r_i(n)$ at time n
- Combined sending rate $R(n)$ is then

$$R(n) = \sum_{i=1}^{N} r_i(n)$$

- This is the total input load on that link

Congestion Control 2

- 1) Design controllers for each $r_i(n)$ so that $R(n)$ converges to some predetermined utilization:

$$\lim_{n \to \infty} R(n) = C(1 - \delta)$$

  - This is called convergence to efficiency
  - If $\delta = 0$, then link utilization is 100% and packet loss is 0% (in some cases 100% utilization might not be desirable)
- 2) Each flow can only rely on its local state
  - This means that the rates of other flows are not known

$$r_i(n) = F_i(r_i(n-1), \ldots, r_i(0), f_i(n))$$

  - Rate r_i does not explicitly depend on r_j for $j \neq i$

Congestion Control 3

- 3) All flows must achieve fairness
  - Define $\Phi(n) = \Phi(r_1(n),\ldots,r_N(n))$ to be some fairness metric
  - Fairness is typically between 0 (maximum unfairness) and 1 (maximum fairness)
  - Then the following must hold:

$$\lim_{n \to \infty} \Phi(n) = 1$$

- 4) The system must be stable
  - When perturbed, fairness $\Phi(n)$ and combined rate $R(n)$ converge back to 1 and $C(1-\delta)$, respectively

Fairness

- We only consider fairness in a single-link network
  - Multi-link fairness is complex and beyond the scope
- Max-min fairness
  - Suppose $r = (r_1, \ldots, r_N)$ is the sequence of flow rates such that $\max(r) > 0$
  - Then the max-min fairness of this sequence is:

$$\Phi(r_1, \ldots, r_N) = \min_{i,j}\left(\frac{r_i}{r_j}\right) = \frac{\min(r)}{\max(r)}$$

- Notice the discontinuous (and non-differentiable) nature of Φ

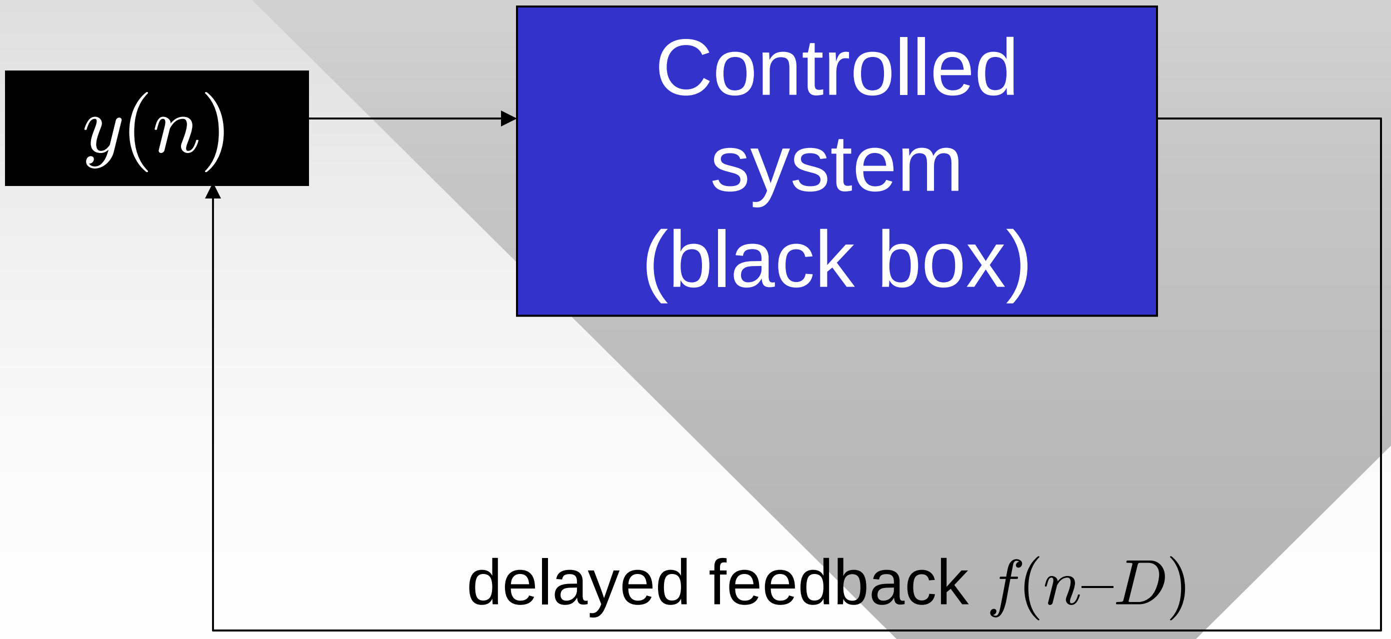
Fairness 2

- Another approach is to look at “majority fairness”
  - The idea is that if the majority of flows are fair among each other, then the fairness index should reflect this situation
  - Majority fairness metrics are also more tractable mathematically
- Chiu and Jain (1984) define the following

$$\Phi(r_1, \ldots, r_N) = \frac{\left(\sum_{i=1}^{N} r_i\right)^2}{N \sum_{i=1}^{N} r_i^2} = \frac{E^2[X]}{E[X^2]} = \frac{E^2[X]}{E^2[X] + Var[X]}$$

Stability

- To understand control systems, one must examine stability properties of the corresponding differential equations
- Recall our generic model:



Stability 2

- Start with several simplifying assumptions
  - Delay D is zero
  - We have only 1 controlled parameter
- We want to design a differential equation that converges to the desired state
- What can be said about the behavior of

$$\frac{dy}{dt} = F(y(t), f(t))$$

  - where $f(t)$ is the feedback (packet loss for example)?

Stability 3

- Questions to ask
  - Does the solution starting in all initial states $y(0)$ converge to a unique point?
  - Does it oscillate around that point?
  - Does it diverge to infinity?
- Definition: point x^* is a *stationary* point of equation:

$$\frac{dx(t)}{dt} = F(x(t))$$

  - if it is a root of function F, i.e., $F(x^*) = 0$
- A system started in x^* stays there

Stability 4

- Discrete systems are often called *maps*
  - The equation is written as $x(n+1) = F(x(n))$
  - The stationary point here is called a fixed point
- Definition: x^* is a fixed point of a discrete map (recurrence) if $F(x^*) = x^*$
  - In other words: x^* is a root of equation $F(x) - x = 0$
- Corollary: if a system does not have stationary points, it either oscillates or diverges
  - Divergence means it tends to infinity in some sense

Stability 5

- Examples
  - Find stationary points of:

$$\frac{dx(t)}{dt} = -(x - C)$$

  - and:

$$\frac{dx(t)}{dt} = -(x^2 + C)$$

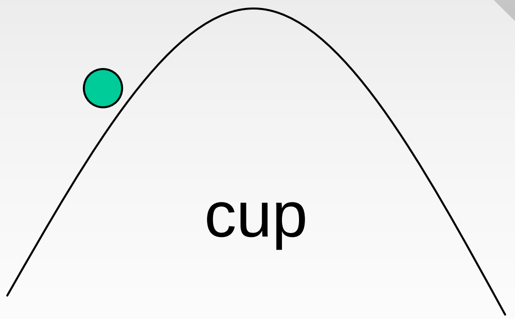
$$x(n+1) = -(x(n)^2 + C)$$

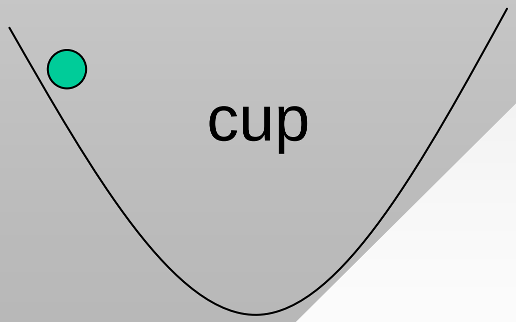
  - and:

$$\begin{cases} \dfrac{dx(t)}{dt} = x - (y^2 + C) \\ \dfrac{dy(t)}{dt} = y^2 - (x^2 - C^2) \end{cases}$$

Stability 6

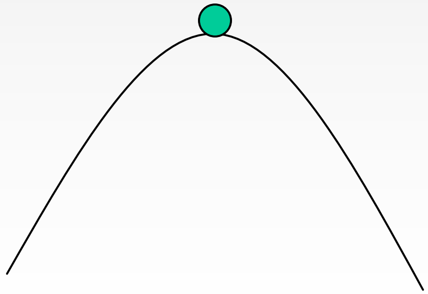
- Is stationarity of a point the end of the story?
  - Are all stationary points equal in their “usefulness”?
- <u>Example</u>: Place a ball inside or outside a cup – how many stationary points do these systems have?

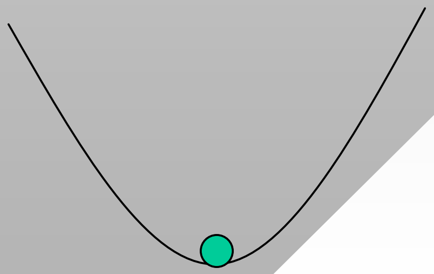




Stability 7

- There is one stationary point in each case
  - Located at the very top (bottom) of each curve
- However, stationarity alone does not tell us what happens to the ball if we perturb the system
  - Hence, we need an additional parameter – stability, which tells us whether the system returns back to the stationary state or not





Stability 8

- Clearly, the system on the right is stable, but not the one on the left
  - In fact, the system on the left diverges to infinity if the ball is lightly touched (system is unstable)
- Stability is desired in all systems one designs
  - We look at several concepts from Lyapunov stability theory
- Definition: $\varphi(x_0,t)$ is called a *flow* (trajectory) of system $dx/dt = F(x,t)$ if it is the solution to the system started in point x_0
- Example: find the flow of

$$\frac{dx(t)}{dt} = -(x-a)$$

- Answer:

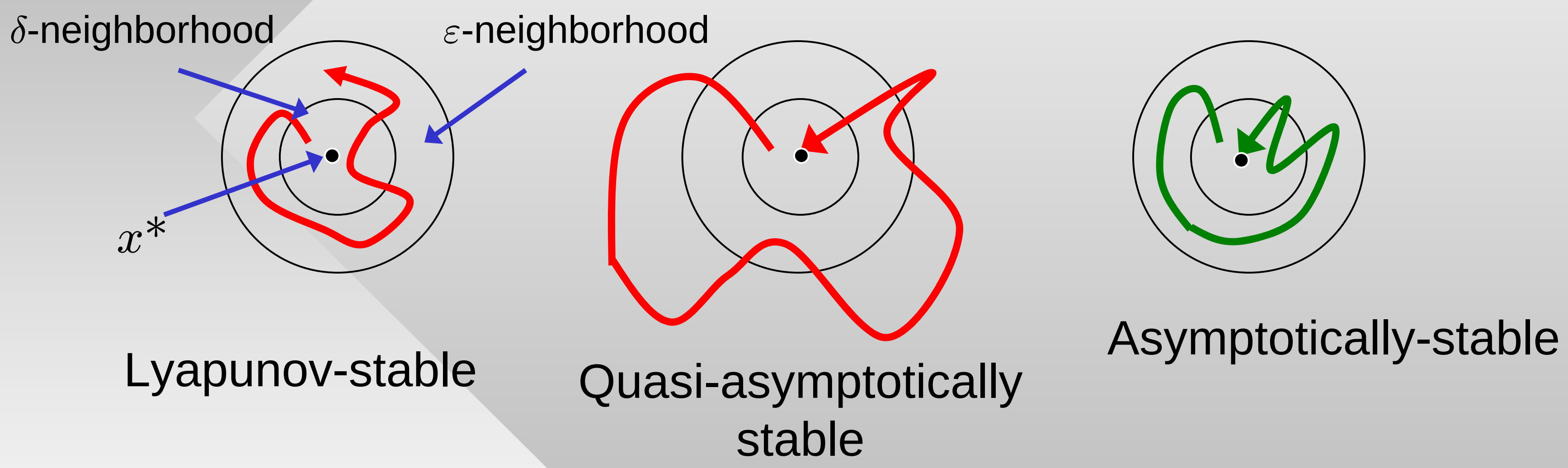
$$\varphi(x_0,t) = (x_0 - a)e^{-t} + a$$

Stability 9

- Definition: point x^* is *Lyapunov-stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_0 - x^*| < \delta$ then $|\varphi(x_0,t) - x^*| < \varepsilon$ for all $t > 0$
- Definition: point x^* is *quasi-asymptotically* stable if there exists $\delta > 0$ such that if $|x_0 - x^*| < \delta$ then $|\varphi(x_0,t) - x^*| \to 0$ as $t \to \infty$
- Definition: point x^* is *asymptotically* stable if it is both Lyapunov and quasi-asymptotically stable
  - Note: asymptotic stability is what we need in practice
- Next consider graphical examples

Stability 10

- Three types of stability:



- Some examples:

$$\dot{x}(t) = -5x \qquad \dot{x}(t) = x(1-x)$$

$$x(n+1) = x(n) \qquad x_{n+1} = x_n + x_n(1-x_n)$$

Stability 11

- Example 1 $\dot{x}(t) = -5x$
  - Analyze stability and solve the equation
  - What are its stationary points?
- Clearly, $x^* = 0$ is the only stationary point
- If you can solve the equation, you can easily prove stability (convergence/divergence)
  - Divide by x and integrate:
  - Converges to 0 from any initial point $x(0)$
  - Asymptotically stable

$$\int \frac{dx}{x} = -5 \int dt$$

$$x(t) = x(0)e^{-5t}$$

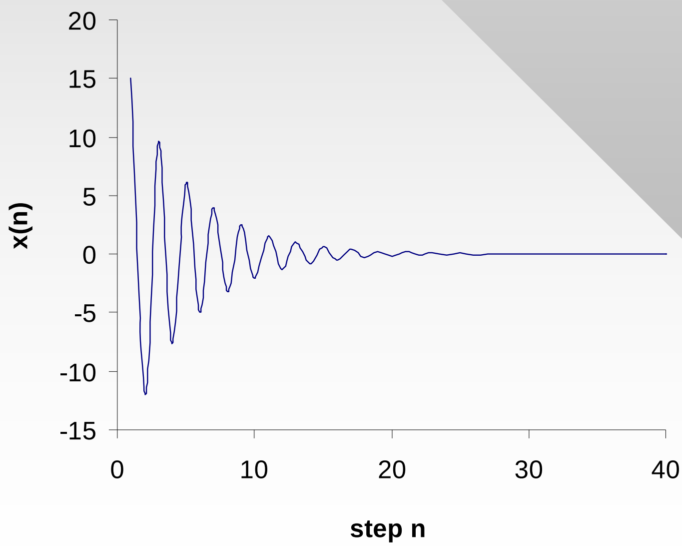
Stability 12

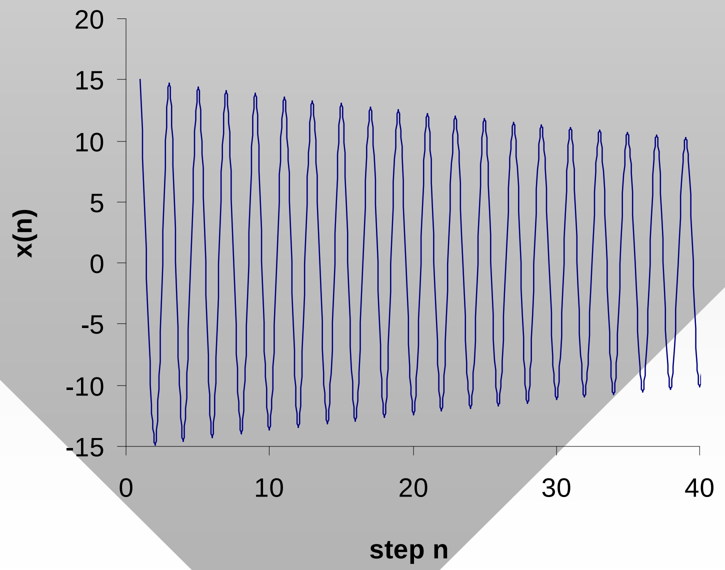
- What is $\varphi(x_0,t)$ for this case?
  - Notice that $\varphi(x_0,t) = x(t) = x_0 e^{-5t}$ is the flow (trajectory) of the system
  - Thus we can show the two necessary conditions:
    1) $x(t) \to x^* = 0$ (quasi-asymptotic stability)
    2) for all ε, choose $\delta = \varepsilon$; then for $|x_0 - x^*| < \delta$, we can say $|x(t) - x^*| < \varepsilon$, for all $t > 0$ (Lyapunov stability)
- This example shows the following lemma
  - Lemma: if the system converges to a stationary point monotonically, it is asymptotically stable
  - Condition is sufficient, but not necessary

Stability 13

- Are there linear systems that are asymptotically stable, but not monotonically convergent?
  - Charts below use $a = 1.8$ and 1.99

$$x(n) = (1 - a)x(n - 1)$$



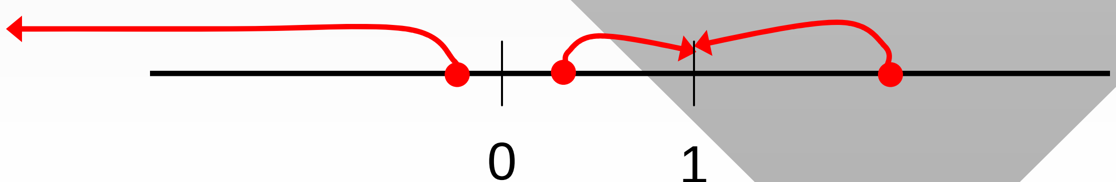


Stability 14

- Example 2: $$x(n+1) = x(n)$$
  - Fixed points?
  - How many?
- Any point $x^* \in \mathcal{R}$ is a fixed point
- Are these points stable?
  - Asymptotically stable? No, since once perturbed, the system does not converge back
  - Lyapunov stable? Yes, since if we start the system in some neighborhood of any x^*, the system stays there

Stability 15

- Example 3: $\dot{x}(t) = x(1-x)$
  - Notice non-linearity in the equation
  - Stationary points?
- There are two points: ${x_1}^* = 0$ and ${x_2}^* = 1$
- Stability?
  - Check by starting the system in a small neighborhood of each stationary point and examining its behavior
  - Point ${x_1}^*$ is not stable, but ${x_2}^*$ is



Stability 16

- Definition: A system is *globally stable* if it is asymptotically stable for any initial state $x(0)$
- Definition: A system is *locally stable* if there exists some neighborhood S of x^* where the system is asymptotically stable for all $x(0) \in S$
  - S is called the region of stability
- For linear systems, local stability implies global
  - For non-linear cases, the two may be different
  - Example 3 is one such case

Stability 17

$$x_{n+1} = x_n + x_n(1 - x_n)$$

- Example 4:
  - Notice that this is the discrete version of example 3
  - Fixed points?
- As before, $x_1* = 0$ and $x_2* = 1$
  - Point x_1* is unstable: small perturbations lead to diverging trajectories (where do they diverge?)
  - Point x_2* is locally stable: follows from the analysis of the continuous version in example 3
- However, x_2* region of stability is not $(0, \infty)$ as was the case for the continuous version
  - To verify, start with $x_0 = 5$

Stability 18

- Rewrite the last equation as

$$x_{n+1} = x_n(2 - x_n)$$

- Informal analysis of stability region S
  - Notice that if $x_0 = 2$, $x(1) = 0$ and then the system is stuck in the unstable stationary point $x_1{}^*$
  - If $x_0 > 2$, then $x(1)$ is strictly negative; recall that once the system goes negative, it diverges to $-\infty$
  - Therefore, $x_0 \geq 2$ does not converge
- Using similar arguments, any point $0 < x_0 < 2$ keeps the system stable

Stability 19

- Formal analysis
  - Define the distance to the fixed point at time n:

$$D_n = |x^* - \varphi(x_0, n)| = |1 - x(n)|$$

- We will show that $D(i+1) < D(i)$ as long as $0 < x(0) < 2$, which is sufficient for asymptotic stability

$$\begin{aligned} D_{i+1} &= |1 - x_{i+1}| = |1 - x_i(2 - x_i)| \\ &= (1 - x_i)^2 = D_i^2 < D_i \end{aligned}$$

  - In fact, convergence is double-exponential (very fast)

$$D_i = (D_0)^{2^i}$$

Wrap-up

- Region of stability is indeed $(0,\ 2)$
  - Much stricter than in the continuous case, which had $(0,+\infty)$
- Several observations
  - Discrete systems are usually more demanding in their requirements for stability, because continuous systems make infinitely small steps and avoid drastic jumps of discrete systems
  - Derivations of stability conditions according to the definition can become tedious even for the simplest systems
- To make life easier, there are several stability-analysis techniques that do not require solving equations or graphical analysis of trajectories
  - We leave these results for next time