CSCE 619-600
Networks and Distributed Processing
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Congestion Control III

Dmitri Loguinov
Texas A&M University

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Agenda

- Linear Stability
  - Systems of equations
  - Linearly independent eigenvalues
  - General case
- Stability of differential equations

Linear Stability

- We first review several results from matrix algebra and how they related to differential equations
- Assume a system of N equations:

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + \mathbf{b}$$

- Vector $\mathbf{x}(t)$ is composed of N individual functions and $\mathbf{b}$ is a vector of constants:

$$\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$$

$$\mathbf{b} = (b_1, \ldots, b_N)$$

Linear Stability 2

- Convenient to shift the variables so that the stationary point is $(0, 0, ..., 0)$
- We generally assume that A is nonsingular
  - In other words, $\det(A) \neq 0$ (there is no linear dependency between the rows) and we have a unique stationary point
- Since the inverse of A exists, we can set:

$$\mathbf{y}(t) = \mathbf{x}(t) + A^{-1}\mathbf{b}$$

  - which leads to

$$\frac{d\mathbf{y}(t)}{dt} = A\mathbf{y}(t)$$

Linear Stability 3

- Definition: λ is an eigenvalue of matrix A if there exists a non-zero vector $\mathbf{v}$ such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

  - Each $\mathbf{v}$ is called an eigenvector
- Each λ has an infinite set of associated eigenvectors
  - Notice that if $A\mathbf{v}=\lambda\mathbf{v}$, then $A(a\mathbf{v})=\lambda(a\mathbf{v})$ for any $a \neq 0$
- Define λ_1, ..., λ_k to be the set of all eigenvalues of A and M_i to be the set of linearly independent eigenvectors corresponding to λ_i
- Theorem: eigenvalues are solutions to the characteristic equation of the system:

$$\det(A - \lambda I) = 0$$

Linear Stability 4

- Proof by contradiction: since $A\mathbf{v}=\lambda\mathbf{v}$, it follows that:

$$(A - \lambda I)\mathbf{v} = 0$$

  - Define $B = A - \lambda I$ and suppose that $\det(B) \neq 0$
  - Then, $\mathbf{v} = B^{-1}0 = 0$ is the only solution to the equation, which contradicts the fact that λ is an eigenvalue
  - Therefore, $\det(B)$ must be zero
- Definition: the characteristic polynomial of A is

$$P(\lambda) = \det(A - \lambda I)$$

- It has degree N and can be written as:

$$P(\lambda) = \alpha_0 \lambda^0 + \ldots + \alpha_N \lambda^N$$

  - for some constants α_i

Linear Stability 5

- Definition: eigenvalue λ_i has algebraic multiplicity m if $(\lambda - \lambda_i)^m$ is the highest power of $(\lambda - \lambda_i)$ that divides the characteristic polynomial
- Expressing $P(\lambda)$ as a product, we can write:

$$P(\lambda) = (\lambda - \lambda_1)^{m_1} \times \ldots \times (\lambda - \lambda_k)^{m_k}, \qquad \sum_{i=1}^{k} m_i = N$$

  - where k is the number of distinct roots of the polynomial and m_i is the algebraic multiplicity of eigenvalue λ_i
- Theorem: the set of eigenvectors corresponding to different λ_i is linearly independent
  - Linear independence means that no vector in the set can be expressed as a linear combination of the other vectors

Linear Stability 6

- Definition: eigenvalue λ_i has geometric multiplicity g_i if it has g_i linearly independent eigenvectors associated with it (i.e., $|M_i| = g_i$)
  - The geometric multiplicity is always no more than the algebraic multiplicity (i.e., $g_i \leq m_i$)
- Example:
  - Consider two matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Linear Stability 7

- Example (continued)
  - The first matrix has two eigenvalues

$$P(\lambda) = (\lambda - 2)(\lambda + 1) \quad \begin{array}{ll} \lambda_1 = 2 & M_1 = \{(2,1)\} \\ \lambda_2 = -1 & M_2 = \{(-1,1)\} \end{array}$$

  - Multiplicity is $m_1 = g_1 = m_2 = g_2 = 1$
- The second one has a single eigenvalue and the corresponding family of vectors is:

$$P(\lambda) = (\lambda - 1)^2 \qquad \lambda_1 = 1 \quad M_1 = \{(0,1)\}$$

  - Thus, $g_1 = 1$, but $m_1 = 2$

Linear Stability 8

- Define matrix M to consist of all linearly independent eigenvectors of the system:

$$M = (\mathbf{v}_1, \ldots, \mathbf{v}_l)$$

  - For each λ_i, we take its independent-vector set M_i
- If A has N different eigenvalues, then $l = N$ and M forms an N-dimensional *basis*
  - In other words, every N-dimensional vector $\mathbf{x}$ can be expressed as a linear combination of $\mathbf{v}_i$

$$\mathbf{x} = \sum_{i=1}^{N} c_i \mathbf{v}_i$$

- We next leverage this fact

Linear Stability 9

- Return to our system of N linear equations:

$$\frac{d\mathbf{y}(t)}{dt} = A\mathbf{y}(t)$$

$$\frac{dy_i(t)}{dt} = a_{i1}y_1(t) + ... + a_{iN}y_N(t)$$

  – How do we determine its stability?
- Suppose that A has N independent eigenvectors
  – The case when $l < N$ is more difficult and will not be examined in detail
- Then, any initial state $\mathbf{y}(0)$ can be expressed as:

$$\mathbf{y}(0) = \sum_{i=1}^{N} c_i\mathbf{v}_i$$

Linear Stability 10

- We conjecture that the solution has the form of:

$$\mathbf{z}(t) = \sum_{i=1}^{N} c_i \mathbf{v}_i e^{\lambda_i t}$$

  - and next prove that it satisfies our system
- Notice that:

$$\frac{d\mathbf{z}(t)}{dt} = \sum_{i=1}^{N} c_i \mathbf{v}_i \lambda_i e^{\lambda_i t}$$

- At the same time:

$$A\mathbf{z}(t) = \sum_{i=1}^{N} c_i (A\mathbf{v}_i) e^{\lambda_i t} = \sum_{i=1}^{N} c_i \lambda_i \mathbf{v}_i e^{\lambda_i t}$$

Linear Stability 11

- Since both expressions are the same, $\mathbf{z}(t)$ is the solution to the original system of equations
- The final step is to prove that $\mathbf{z}(0) = \mathbf{y}(0)$:

$$\mathbf{z}(0) = \sum_{i=1}^{N} c_i \mathbf{v}_i = \mathbf{y}(0)$$

- In a more general case when matrix M is singular, each of c_i is a polynomial function of t and the solution is given by:

$$\mathbf{z}(t) = \sum_{i=1}^{N} c_i(t) \mathbf{v}_i e^{\lambda_i t}$$

Linear Stability 12

- Define $r = \max \mathrm{Re}(\lambda_i)$ (largest real part of any eigenvalue of matrix A)
- Theorem: if 1) $r < 0$, system $d\mathbf{y}/dt = A\mathbf{y}$ is asymptotically stable
- 2) $r > 0$, the system is unstable and diverges to infinity
- 3) $r = 0$:
  - System is marginally (Lyapunov) stable (oscillates or stays constant) if M is nonsingular (i.e., $l = N$)
  - System diverges otherwise (i.e., $l < N$)
- Proof: by examining $\mathbf{z}(t)$ for all cases individually
- Note: for linear systems, stability is always considered in the global sense (i.e., for any initial state $\mathbf{y}(0)$)

Linear Stability 13

- Example:
  - Derive stability of the following system:
  - Write:

$$\begin{cases} \dot{x} = -\beta_1 y \\ \dot{y} = -\beta_2 x \end{cases}$$

$$A = \begin{bmatrix} 0 & -\beta_1 \\ -\beta_2 & 0 \end{bmatrix}, A - \lambda I = \begin{bmatrix} -\lambda & -\beta_1 \\ -\beta_2 & -\lambda \end{bmatrix}$$

  - And find the roots of the characteristic polynomial:

$$\lambda^2 - \beta_1\beta_2 = 0$$

Linear Stability 14

- Example (continued)
  - The two eigenvalues are:

$$\begin{cases} \lambda_1 = +\sqrt{\beta_1\beta_2} \\ \lambda_2 = -\sqrt{\beta_1\beta_2} \end{cases}$$

- First suppose that $\beta_1\beta_2 < 0$
  - Then both roots have a zero real part, which leads to marginal stability (since there are two different eigenvalues and therefore $l = N = 2$)
- Next suppose that $\beta_1\beta_2 > 0$
  - Then $\lambda_1 > 0$, $\lambda_2 < 0$ and the system is unstable
- The final case is $\beta_1\beta_2 = 0$
  - If $\beta_1 = \beta_2 = 0$, $M =$ {$(1{,}0)$, $(0{,}1)$} and $l = N$, which means the system is Lyapunov stable since it just stays in the initial state ($dx/dt = 0$, $dy/dt = 0$)

Linear Stability 15

- The final two cases: $\beta_1 \neq 0, \beta_2 = 0$ and $\beta_1 = 0, \beta_2 \neq 0$ are more tricky since $l < N$

- Assume for example that $\beta_1 \neq 0, \beta_2 = 0$
  - Then we have:

$$\begin{cases} \dot{x} = -\beta_1 y \\ \dot{y} = 0 \end{cases}$$

  - Matrix A is given by:

$$A = \begin{pmatrix} 0 & -\beta_1 \\ 0 & 0 \end{pmatrix}$$

- Single eigenvalue $\lambda = 0$, whose set of eigenvectors is $M = \{(1, 0)\}$ and thus $l = 1 < N$

Linear Stability 16

- The system is thus unstable and tends to positive or negative infinity depending on the initial state
- This can be verified by directly solving the system:

$$\begin{cases} x(t) = x(0) - \beta_1 y(0) t \\ y(t) = y(0) \end{cases}$$

- The only stable trajectory starts in $y(0) = 0$