CSCE 619-600
Networks and Distributed Processing
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Congestion Control IV

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Agenda

- Discrete Linear Stability
- Non-linear Stability
  - Linearization
  - Spectral (eigenvalue) methods
- Delayed Stability
  - Modern control theory

Linear Stability 17

- Similar observations apply to discrete cases
- A discrete system can be written as:

$$\mathbf{y}(n+1) = A\mathbf{y}(n)$$

- The solution to the above is given by:

$$\mathbf{z}(n) = \sum_{i=1}^{N} c_i \mathbf{v}_i \lambda_i^n$$

- Which can be verified directly:

$$A\mathbf{z}(n) = \sum_{i=1}^{N} c_i A \mathbf{v}_i \lambda_i^n = \sum_{i=1}^{N} c_i \mathbf{v}_i \lambda_i^{n+1} = \mathbf{z}(n+1)$$

Linear Stability 18

- Definition: spectral radius ρ of A is the maximum absolute value of its eigenvalues:

$$\rho = \max_i |\lambda_i|$$

- Theorem: 1) if the spectral radius of A is strictly less than 1, the system $\mathbf{y}(n+1) = A\mathbf{y}(n)$ is asymptotically stable
- 2) if $\rho > 1$, then the system is unstable
- 3) borderline case $\rho = 1$ leads to Lyapunov stability when $l = N$ and instability otherwise
  - Proof follows from the shape of $\mathbf{z}(n)$

Linear Stability 19

- Example
  - Consider a system

$$\begin{cases} x_1(n) = 2x_1(n-1) - 6x_2(n-1) \\ x_2(n) = 4x_2(n-1) - 3x_1(n-1) \end{cases}$$

  - With matrix A:

$$A = \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

  - and characteristic equation:

$$(2-\lambda)(4-\lambda) - 18 = \lambda^2 - 6\lambda - 10 = 0$$

- Unstable regardless of the initial point:

$$\rho = 3 + \sqrt{19} > 1$$

Linear Stability 20

- Notice that a mix of stable and unstable eigenvalues results in some trajectories converging and others diverging:

$$\mathbf{z}(n) = \sum_{i=1}^{N} c_i \mathbf{v}_i \lambda_i^n$$

- Depending on c_i (i.e., $\mathbf{y}(0)$), the unstable eigenvalues may be multiplied by a zero and taken out of $\mathbf{z}(n)$
- Thus:
  - Stable system: converges for any initial state
  - Unstable: diverges for all but a few initial states
  - Marginally stable: oscillates for all but a few states

Non-Linear Stability

- The situation with non-linear functions is more complicated since one almost never finds a closed-form solution
  - Consider a generic system of N non-linear equations

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t))$$

$$\mathbf{F}(\mathbf{x}(t)) = [F_1(\mathbf{x}), ..., F_N(\mathbf{x})]^T$$

- Suppose functions $F_i(\mathbf{x})$ are differentiable and reasonably smooth
  - For example:

$$\begin{cases} \dot{x} = \sin(y) \\ \dot{y} = \cos(x) \end{cases}$$

Non-Linear Stability 2

- In this case, $F_1(x,y) = \sin(y)$, $F_2(x,y) = \cos(x)$
  - Stationary points are $(\pi/2+\pi k,\ \pi m)$ for all integer k,m
- Definition: the Jacobian of a non-linear system of equations is given by:

$$J = \left[\begin{array}{ccc} \frac{\partial F_1(\mathbf{x}(t))}{\partial x_1} & \dots & \frac{\partial F_1(\mathbf{x}(t))}{\partial x_N} \\ \dots & \dots & \dots \\ \frac{\partial F_N(\mathbf{x}(t))}{\partial x_1} & \dots & \frac{\partial F_N(\mathbf{x}(t))}{\partial x_N} \end{array} \right] \Bigg|_{\mathbf{x}^*}$$

- Each row is a set of partial derivatives evaluated in the stationary point (vector) $\mathbf{x}^*$ of the system

Non-Linear Stability 3

$$J = \left[\begin{array}{ccc} \frac{\partial F_1(\mathbf{x}(t))}{\partial x_1} & \cdots & \frac{\partial F_1(\mathbf{x}(t))}{\partial x_N} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_N(\mathbf{x}(t))}{\partial x_1} & \cdots & \frac{\partial F_N(\mathbf{x}(t))}{\partial x_N} \end{array}\right]\Bigg|_{\mathbf{x}^*}$$

- Definition: linearization is a process of replacing a non-linear system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ with a linear system $d\mathbf{x}/dt = J\mathbf{x}$ in a small neighborhood of a given stationary point $\mathbf{x}^*$:

$$\frac{d\mathbf{x}(t)}{dt} = J\mathbf{x}(t)$$

- Example: linearize this system:

$$\begin{cases} \dot{x} = \sin(y) \\ \dot{y} = \cos(x) \end{cases}$$

- Pick a point of interest, say $\mathbf{x}^* = (\pi/2,\, 0)$
  - Next, compute the Jacobian and evaluate it in $\mathbf{x}^*$

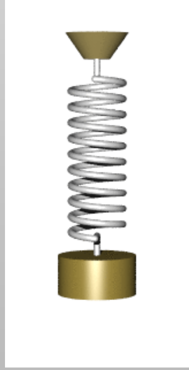
Non-Linear Stability 4

- The Jacobian:

$$J = \begin{bmatrix} 0 & \cos(y) \\ -\sin(x) & 0 \end{bmatrix}$$

- Substitute $(x,\, y) = \mathbf{x}^* = (\pi/2,\, 0)$ into the matrix:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



- The linearized equation is:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

- The harmonic oscillator above is not asymptotically stable (concentric circles)

Non-Linear Stability 5

- Stable Manifold Theorem: A non-linear system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ is asymptotically stable (unstable) in a stationary point $\mathbf{x}^*$ if the linearized system $d\mathbf{x}/dt = J\mathbf{x}$ is asymptotically stable (unstable) at the origin
  - "At the origin" means in $\mathbf{x} = (0, 0, \ldots, 0)$
- This is an important theorem for non-linear systems
  - It says that in the small neighborhood of $\mathbf{x}^*$, non-linear systems behave the same as their linear counterparts
- If $d\mathbf{x}/dt = J\mathbf{x}$ is marginally stable, then nothing useful can be said about $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$

Non-Linear Stability 6

- Recall an old example:

$$\frac{dx}{dt} = x(1-x)$$

- Jacobian $J = 1 - 2x*$
- First analyze $x* = 0$
  - The linear system is $dx/dt = x$
  - The only eigenvalue is $\lambda = J = 1$, the system is unstable
- Second analyze $x* = 1$
  - The linear system is $dx/dt = -x$
  - We get $\lambda = -1 < 0$, the system is stable

Non-Linear Stability 7

- Another example:
  - How many stationary points?

$$\begin{cases} \dot{x} = x(1-y) \\ \dot{y} = y(1-x) \end{cases}$$

- There are two:
  - $(0,0)$ and $(1,1)$
- Compute the Jacobian

$$J = \begin{bmatrix} 1-y & -x \\ -y & 1-x \end{bmatrix}$$

- Now analyze $\mathbf{x}^* = (1,\, 1)$:

$$J(1,1) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Non-Linear Stability 8

- Characteristic equation

$$\lambda^2 - 1 = 0$$

  - Has one positive root, the system is unstable
- In $\mathbf{x}^* = (0,\, 0)$, we have:

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The only root is $\lambda = 1$ (algebraic multiplicity 2)
  - System unstable since $\lambda > 0$

Delayed Stability

- It is common to find delayed equations in congestion control
  - Simply because the feedback is delayed
  - One such example in homework #6
- In many of these cases, it is impossible to directly apply the methods from the previous lecture
- Two approaches to study delayed stability
  - Use the z-transform (classical control theory, 1940s)
  - Convert the system to an undelayed version using matrix algebra (modern control theory, 1960s)

Delayed Stability 2

- We first examine the modern control theory approach
- Assume a single equation for simplicity:

$$x(n) = a_1 x(n-1) + a_2 x(n-2) + \ldots + a_D x(n-D)$$

  - Constant D is fixed (it can also be time-varying, but that's more complicated)
- Let us introduce D–1 new functions:

$$\begin{cases} u_1(n) = x(n-1) \\ u_2(n) = x(n-2) \\ \ldots \\ u_{D-1}(n) = x(n-D+1) \end{cases}$$

Delayed Stability 3

- We then obtain an undelayed system of the following shape:

$$\mathbf{y}(n) = A\mathbf{y}(n-1)$$

  - where $\mathbf{y}(n)$ is a column of individual functions:

$$\mathbf{y}(n) = (x(n), u_1(n), \ldots, u_{D-1}(n))^T$$

  - Note here that T stands for “transpose”

- The trick is to notice that we can write:

$$\begin{cases} u_1(n) = x(n-1) \\ u_2(n) = x(n-2) \\ \ldots \\ u_{D-1}(n) = x(n-D+1) \end{cases} \Leftrightarrow \begin{cases} u_1(n) = x(n-1) \\ u_2(n) = u_1(n-1) \\ \ldots \\ u_{D-1}(n) = u_{D-2}(n-1) \end{cases}$$

Delayed Stability 4

- Re-writing the original equation, we get:

$$x(n) = a_1 x(n-1) + a_2 u_1(n-1) + \ldots + a_D u_{D-1}(n-1)$$

- And matrix A has the following shape:

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_{D-1} & a_D \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & & & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

- The system is stable if and only if the spectral radius of A is less than 1

Delayed Stability 5

- Example:
  - Consider the following equation, where β is some constant gain parameter

$$x(n) = \beta x(n-1) + 2x(n-2)$$

- Introducing $u(n)$, we get:

$$\begin{cases} x(n) = \beta x(n-1) + 2u(n-1) \\ u(n) = x(n-1) \end{cases}$$

- Which can also be expressed as:

$$\mathbf{y}(n) = A\mathbf{y}(n-1) \qquad A = \begin{pmatrix} \beta & 2 \\ 1 & 0 \end{pmatrix}$$

Delayed Stability 6

- Example (continued)
  - Which produces the characteristic polynomial:

$$P(\lambda) = -(\beta - \lambda)\lambda - 2 = 0$$

  - In other words:

$$\lambda^2 - \beta\lambda - 2 = 0$$

- Caveat: when backward delay D is large, this method leads to huge $D \times D$ matrices and is difficult to solve in closed-form
  - Example of this:

$$x(n) = \beta x(n-1) + 2x(n-20)$$