CSCE 619-600: Computer Networks
Homework 2 due February 16, 2017

1 Purpose

This homework reinforces understanding of basic probability theory and studies properties of stochastic processes.

2 Description

The first part requires solving certain problems from Wolff's book, while the second part calls for simulations and related analysis.

2.1 Problems from the Book

Solve the following problems: 1-30 on page 48, 2-19 on page 132, and 2-43 on page 136.

2.2 Basic Wealth Evolution Model

Each person i starts with $X_i(0) = 1$ dollar in their bank. Each year, they invest in the stock market, earn a certain salary, spend random amounts of money on various activities, etc. Assume that the evolution of their wealth can be modeled by:

$$X_i(n) = X_i(n-1) + w_i(n)X_i(n-1), \tag{1}$$

where $w_i(n)$ is a uniform variable in $[0, 1]$. Note that $w_i(n)$ and $w_j(k)$ are independent of each other for all i, j, n, k.

1. Derive the distribution of wealth in the society after n years. In other words, obtain the CDF or PDF of $X_i(n)$.
2. Using $n = 10$ and $K = 10$M users, run simulations to confirm your derivations. Plot the histogram of set $\{X_1(n), \ldots, X_K(n)\}$ and your model. Observe and discuss results. Figure 1 shows an idea of what your histogram should look like.

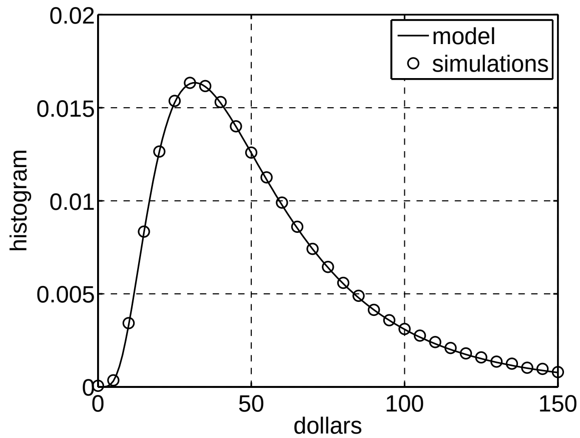

Figure 1: Distribution of wealth after 10 years (bin size $\Delta = 1$ dollar).

3. Using your simulation data, record two metrics – the ratio γ of the maximum wealth to the average and the ratio δ between the upper and lower 1% of the population.

Hints: 1) notice that $X_i(n)$ is a product of n variables; apply a transformation that reduces it to a summation; 2) assume n is sufficiently large to invoke the Central Limit Theorem; 3) use the integrals below to compute the mean and variance of the resulting Gaussian distribution; and 4) Excel has a built-in function for the CDF of $X_i(n)$, which you can use to obtain the values for each bin. Note that Figure 2.2 is *not* a PDF, but a histogram. If bin size is Δ, then the probability to observe a sample in bin $[i\Delta, (i+1)\Delta]$ is $F((i+1)\Delta) - F(i\Delta)$, where $F(x)$ is the CDF of the random variable.

2.3 Extended Wealth Evolution

Next, notice that not all people started their bank accounts at the same time. Thus, assume that n is also a random variable uniformly distributed in say [10, 40], which represents 10 to 40 years of accumulating wealth up to the current moment. To prevent unconditional growth of your wealth each year and allow for possible bankruptcies, market fluctuations, and various over-spending activity, finally assume that w is uniform in $[-0.3, 0.7]$ (i.e., with probability 30%, a person's wealth will decrease in a given year and with probability 70% it will increase).

1. Let $K = 10$M be the number of people in your society. In the simulation, execute the following process for each person i: 1) generate a random number of years n_i; 2) run process $X_i(n)$ from time 0 to n_i, where $X_i(0) = 1$; and 3) record $X_i(n_i)$ into some set S. Using techniques from section

2.2, plot the histogram of random variable Y whose values are given by set $S = \{X_1(n_1), \ldots, X_K(n_K)\}$.

2. Analyze the obtained distribution by finding the most appropriate representation of the graph (e.g., log-linear, log-log, linear-log) that *clearly* suggests its shape and then use curve-fitting to find a model for it. Since the tail of the histogram is a scaled version of the PDF tail, obtain parameters of the distribution that models Y.

3. Examine whether this new society has a more equal or more skewed distribution of wealth in comparison to the model of Section 2.2. Use the same metrics γ and δ.

3 Integrals

You may find the following useful:

$$\int \log(1+x)dx = (1+x)\log(1+x) - 1 - x, \quad (2)$$

$$\int_0^1 (\log(1+x) - a)^2 dx = 2\log^2 2 + a^2 - 4\log 2 + 2a + 2 - 4a\log 2. \quad (3)$$