CPSC 619-600: Computer Networks
Homework 6 due May 5, 2017

1 Purpose

This homework studies linear and non-linear control systems in the presence of delays in the feedback.

2 Description

2.1 Vehicle Cruise Control

2.1.1 Proportional Control

Assume a vehicle with a speed sensor that provides feedback to the cruise control module about the current speed $v(t)$ of the vehicle. In the first case, we study the *proportional* controller in which the actions are proportional to the error between the desired outcome and the current feedback. Assuming no delay in the feedback, the system can be described by a set of equations:

$$\begin{cases} \dot{v}(t) &= y(t) - a_f \\ \dot{y}(t) &= -\kappa e(t) = -\kappa(v(t) - v_d) \end{cases}, \tag{1}$$

where a_f is the combined resistance of the wind, friction, and any possible hills. The acceleration $y(t)$ of the vehicle is the controlled parameter, $v(t)$ is the observed feedback, and your cruise control has no knowledge of a_f. Derive all stationary points (y^*, v^*) of the system. Examine asymptotic stability of the above system. Is it stable? Does it diverge? Does it oscillate?

Next, investigate two discrete versions of (1). One possible version is:

$$\begin{cases} v(n+1) &= v(n) + y(n) - a_f \\ y(n+1) &= y(n) - \kappa(v(n+1) - v_d) \end{cases}, \tag{2}$$

in which feedback $v(n+1)$ is available immediately. What is the stability of (2)? How does κ affect oscillations? Study $\kappa = 3.99$. Why does $\kappa \geq 4$ or $\kappa \leq 0$ make the system divergent? What about $0 < \kappa < 4$? Plot the trajectory of the system $(y(n), v(n))$ and the individual curves $y(n)$, $v(n)$, and explain your observations (i.e., how does the car behave?). Hint: to derive stability conditions, use the z-transform.

Next, study a more realistic system in which at time $n+1$ only the previous speed $v(n)$ is available to the cruise control:

$$\begin{cases} v(n+1) &= v(n) + y(n) - a_f \\ y(n+1) &= y(n) - \kappa(v(n) - v_d) \end{cases} . \tag{3}$$

What is the stability of this system and how does κ affect it? Derive the spectral radius ρ of the corresponding matrix and examine the possibility of stabilizing the control using κ. Plot simulation results to confirm your derivations.

2.1.2 Proportional-Derivative Control

To improve the behavior of (1), we use a *proportional-derivative* (PD) controller:

$$\begin{cases} \dot{v}(t) &= y(t) - a_f \\ \dot{y}(t) &= -\kappa_1 e(t) - \kappa_2 \dot{e}(t) \end{cases} , \tag{4}$$

where $e(t) = v(t) - v_d$ is the error term and $\dot{e}(t)$ is its derivative. The intuition here is that we need to start reducing the acceleration not *after* $v(t)$ has crossed v_d, but when we are *approaching* it. In order to understand this system, convert it to a discrete version and study it in simulations:

$$\begin{cases} v(n+1) &= v(n) + y(n) - a_f \\ y(n+1) &= y(n) - \kappa_1(v(n) - v_d) - \kappa_2(v(n) - v(n-1)) \end{cases} . \tag{5}$$

Notice that the system uses feedback that is $D = 1$ and $D = 2$ time units in the past to decide its current state. Derive the stationary points of (5) and use the z-transform to deduce stability conditions of this system. Find such values of κ_1 and κ_2 that guarantee stability. Experimentally find two cases – one in which $v(t)$ converges *with* oscillations and one *without*.

2.2 Kelly Congestion Control

Assume that $x(n)$ is the sending rate of a flow at time n. Then, Kelly controls for a single-flow network can be written as:

$$\begin{cases} x(n+1) &= x(n) + \alpha - \beta x(n-D)p(n-D) \\ p(n) &= \dfrac{x(n) - C}{x(n)} \end{cases} , \tag{6}$$

where $p(n)$ is the packet loss at time n, D is the delay in the feedback, $C = 1,000$ is the capacity of the bottleneck link, and $\alpha = 20$ is some additive constant. Note that $p(n)$ can be negative when $x(n) < C$, which signals the flow to aggressively increase its rate. Derive stationary points x^* and p^* and the general condition for this system to be stable. Assuming $\beta = 0.5$, show using your condition that $0 \leq D \leq 2$ keeps the system stable, but $D \geq 3$ makes it unstable. Plot simulations for each case. Hint: to show this result, use Matlab to solve the $(D+1)$-st order z-transform equation for all D between 1 and 100 and plot the spectral radius $\rho(D)$ as a function of D. Show that $\rho(D)$ intersects 1 somewhere between 2 and 3.

Next, examine an N-flow system with $D = 0$:

$$\begin{cases} x_i(n+1) &= x_i(n) + \alpha - \beta x_i(n)p(n) \quad 1 \leq i \leq N \\ p(n) &= \dfrac{\sum_{j=1}^{N} x_j(n) - C}{\sum_{j=1}^{N} x_j(n)} \end{cases}, \tag{7}$$

where i is the flow number. Run simulations for $N = 2, C = 1,000, \alpha = 20, \beta = 0.5$ and plot the resulting rates $x_1(n)$ and $x_2(n)$. Do they converge to fairness? Experiment with several initial points $(x_1(0), x_2(0))$ and study convergence. Is the system stable? Does it oscillate? Examine two additional cases $\beta = 1.8$ and $\beta = 2.1$.

Using linearization, obtain the Jacobian matrix of system (7). Derive its eigenvalues and deduce stability conditions on β. Further, show that when $1 \leq \beta < 2$, the system is stable, but oscillates. Hint: you will obtain a very simple circulant matrix in which each row i is a shifted version of the previous row:

$$J = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix}. \tag{8}$$

The eigenvalues of (8) are $\lambda_1 = a - b$ and $\lambda_2 = a + (N-1)b$.