On Delay-Independent Diagonal Stability of Max-Min Congestion Control

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Agenda

• Introduction
  — Modeling of Internet congestion control
  — Current stability results

• Main results

• Applications
  — Delay-independent stable matrices
  — Stability of Max-min Kelly Control (MKC)

• Wrap-up
Max-Min Congestion Control

- Many existing congestion control protocols, such as XCP, RCP, MaxNet, MKC, VCP, and JetMax, are max-min methods.

- In max-min congestion control, each user $i$ calculates its sending rate $x_i(n)$ based on feedback $p_i(n)$ generated by the most-congested link.

- Network feedback is subject to delays, which are not only heterogeneous but also directional.
Feedback Delay

\[ x_i(n) = f_i(p_i(n - D_i^\rightarrow)) \]

\[ p_i(n) = g\left(\sum_j x_j(n - D_j^\rightarrow)\right) \]

\[ D_i^\rightarrow + D_i^\leftarrow = D_i \]

RTT

sender\_i

bottleneck router

receiver\_i
Linearized Max-Min Congestion Control

- Then, the closed-form control equation is

\[ x_i(n) = f_i\left( g\left( \sum_j x_j(n - D_j^\rightarrow - D_i^\leftarrow) \right) \right) \]

- Let \( x^* \) be the equilibrium point of the system, then the linearized system model becomes

\[ x_i(n) = \sum_j a_{ij} x_j(n - D_j^\rightarrow - D_i^\leftarrow) \quad (*) \]

\[ a_{ij} = \frac{\partial f_i}{\partial x_j} \bigg|_{x^*} \]
Equivalent System Model

• Consider following linear system

\[ x_i(n) = \sum_j a_{ij} x_j(n - D_i) \quad (+) \]

• Lemma 1: System (*) is stable under all heterogeneous directional delays \( D_i^\rightarrow \) and \( D_i^\leftarrow \) if and only if system (+) is stable under all round-trip delays \( D_i \)

• System (+) has a simpler shape than (*), so we only consider stability of (+) in the rest of the talk
Assume that the Jacobian matrix $A$ does not involve any delay.

**Definition 1**: We call a system stable independent of delay if its stability condition does not depend on delays.

Clearly, system $(+)$ under zero delay is stable if and only if $\rho(A) < 1$.

Consider $(+)$ under arbitrary delay $D_{ij}$:

$$x_i(n) = \sum_j a_{ij} x_j(n - D_{ij}),$$

which is proved to be stable if and only if $\rho(|A|) < 1$. 
Current Stability Results II

arbitrary delay $D_{ij}$ stable iff $\rho(|A|) < 1$

no delay, stable iff $\rho(A) < 1$

diagonal delay $D_i$ stability condition?

stable under diagonal delay $D_i$ but with $\rho(|A|) > 1$
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Induced Matrix Norm

• Definition 2: The **induced matrix norm** \( \| \cdot \| \) of a given vector norm \( \| \cdot \| \) is defined as follows:

\[
\| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|}
\]

• Examples
  
  – Spectral norm \( \| A \|_2 = \sqrt{\rho(A^*A)} \) (where \( A^* \) is the conjugate transpose of \( A \)) is induced by the \( L^2 \) vector norm
  
  – Maximum absolute column sum norm
  \[
  \| A \|_1 = \max_j \sum_{i=1}^{N} |a_{ij}| \]
  is induced by the \( L^1 \) vector norm
  
  – Maximum absolute row sum norm
  \[
  \| A \|_1 = \max_i \sum_{j=1}^{N} |a_{ij}| \]
  is induced by the \( L^\infty \) vector norm
Extending A Previous Result

- It is proved by Zhang et al. (SIGCOMM04) system (+) is stable if $A$ is symmetric and $\rho(A) < 1$
- However, this result is very restrictive
- Utilizing induced matrix norms, we can obtain an alternative proof of this result and have the following observation
- Corollary 1: System (+) is stable for all delays $D_i$ if $\|A\|_2 < 1$
- Clearly, this condition is tighter (i.e., less restrictive) than the previous result
Verification of Corollary 1

• Matlab simulations
  — Generate 3000 random two-by-two matrices and plot \((x, y)\) where \(x = \rho(A)\) and \(y = \|A\|_2\) of stable and unstable matrices on a 2-D plane

• Thus, Corollary 1 is a sufficient but not necessary
Tighter Sufficient Conditions I

- **Definition 3**: A vector norm $\| \cdot \|$ is monotonic if for all $x, y$ in $\mathbb{R}^n$ such that $\|x\| \leq \|y\|$, it follows $\|x\| \leq \|y\|

- **Theorem 1**: If there exists a monotonic vector norm $\|\cdot\|_\alpha$ such that the induced matrix norm $\|A\|_\alpha < 1$, system $(+)$ is stable regardless of delay $D_i$

- Monotonic norms can be generated using the following result

- **Theorem 2**: Matrix norm $\|A\|_2^w = \|WAW^{-1}\|_2$ for any non-singular diagonal matrix $W = \text{diag}(w)$ is a monotonic induced matrix norm
Tighter Sufficient Conditions II

- **Corollary 2**: System $(+)$ is stable for all delays $D_i$ if
  \[ ||A||_s = \inf_{W \in \mathcal{P}^*} ||WA W^{-1}||_2 < 1 \]
  where $\mathcal{P}^*$ is the set of all positive diagonal matrices

- **Theorem 3**: For any matrix $A$, we have $||A||_s \leq \rho(|A|)$

- Therefore, Corollary 2 does not hold for arbitrary $D_{ij}$. But, is it tight for $D_i$?
Verification of Corollary 2

- We next use Matlab simulations to verify Corollary 2
  - Generate 10000 random two-by-two matrices and plot 3535 stable/unstable matrices

- Conjecture: Condition in Corollary 2 is both sufficient and necessary for $\rho(A) + \rho(\hat{A})$ to be stable under any delay $D_i$.
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Delay-Independent Stable Matrices I

- $\mathcal{N}$: normal matrices, $A$ is in $\mathcal{N}$ if $AA^* = A^*A$
- **Definition 4**: Matrix $A$ is *diagonally similar* to $B$ if there exists a non-singular diagonal matrix $W$ such that $WAW^{-1} = B$
- $\mathcal{DN}$: the set of matrices that are diagonally similar to $\mathcal{N}$
- $\mathcal{P}$: the set of *non-negative/non-positive* matrices
- $\mathcal{DP}$: the set of matrices that are diagonally similar to $\mathcal{P}$
- $\mathcal{R}$: the set of *radial matrices*, $A$ is in $\mathcal{R}$ if $\|A\|_2 = \rho(A)$
- $\mathcal{DR}$: the set of matrices that are diagonal similar to $\mathcal{R}$
• **Theorem 4**: The following matrices are stable under all diagonal delay $D_i$ if and only if $\rho(A) < 1$: $\mathcal{N}$, $\mathcal{DN}$, $\mathcal{P}$, $\mathcal{DP}$, $\mathcal{R}$, $\mathcal{DR}$

• These matrix classes satisfy the following relationship, where $A \rightarrow B$ denotes $A \subset B$

```
\mathcal{N} \leftarrow \mathcal{DN} \rightarrow \mathcal{DR} \leftarrow \mathcal{DP} \leftarrow \mathcal{P}
```

• $\mathcal{DR}$ is the largest class of matrices that are stable under all diagonal delay $D_i$ if and only if $\rho(A) < 1$
Application to Max-min Kelly Control

• MKC end-user equation:

\[ x_i(n) = (1 - \beta p_i(n - D_i^-))x_i(n - D_i) + \alpha \]

sending rate of user \( i \) feedback \( p_i(n) = g\left(\sum_j x_j(n - D_j^-)\right) \)

• Stability of MKC under homogeneous parameters \( \alpha \) and \( \beta \) has been proved

• Theorem 5: Single-link MKC with heterogeneous \( \alpha_i \) and \( \beta_i \) is stable under all diagonal delay \( D_i \) if

\[ 0 < \beta_i(p^* + \sum_{i=1}^{N} x_i^*p_i') < 2, \quad i = 1, \ldots, N \]

where \( x_i^* \) and \( p^* \) are stationary points of \( x_i(n) \) and \( p(n) \)
Wrap-up

• In this paper, we studied stability of max-min congestion control systems under diagonal delays.

• Our results improved the understanding of delay-independent stability from the requirement that $\rho(A) < 1$ and $A$ is symmetric to the simple condition that $\|A\|_s < 1$.

• Simulations suggest that $\|A\|_s < 1$ is also a necessary condition.

• The obtained results are of broader interest and apply to any system that can be modeled by (+) or (*)
