On Delay-Independent Diagonal Stability of Max-Min Congestion Control

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Agenda

- Introduction
  - Modeling of Internet congestion control
  - Current stability results
- Main results
- Applications
  - Delay-independent stable matrices
  - Stability of Max-min Kelly Control (MKC)
- Wrap-up

Max-Min Congestion Control

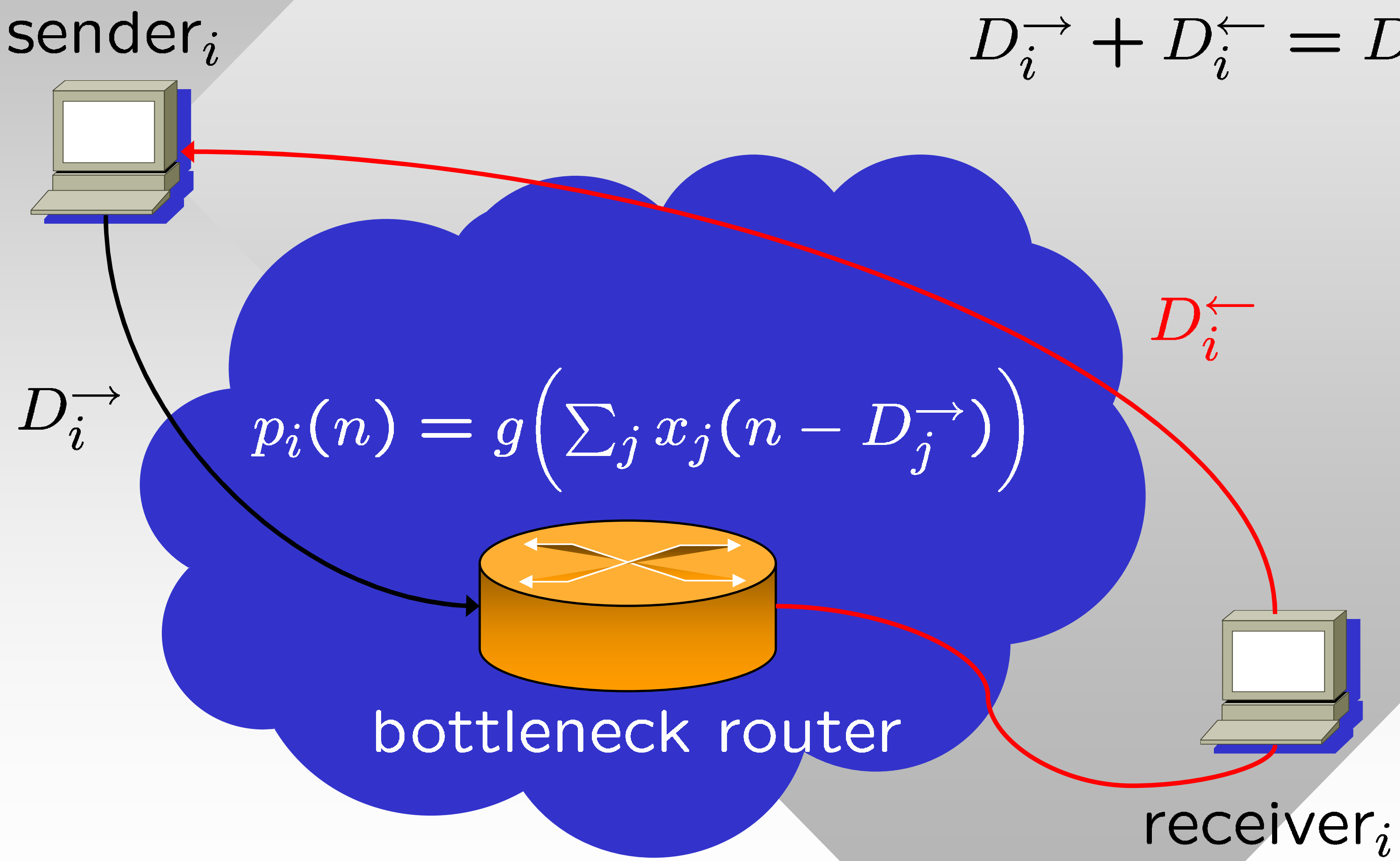
- Many existing congestion control protocols , such as XCP, RCP, MaxNet, MKC, VCP, and JetMax, are max-min methods
- In max-min congestion control, each user i calculates its sending rate $x_i(n)$ based on feedback $p_i(n)$ generated by the most-congested link
- Network feedback is subject to delays, which are not only heterogeneous but also directional

Feedback Delay

$$x_i(n) = f_i(p_i(n - D_i^{\leftarrow}))$$

RTT

$$D_i^{\rightarrow} + D_i^{\leftarrow} = D_i$$



Linearized Max-Min Congestion Control

- Then, the closed-form control equation is

$$x_i(n) = f_i\Big(g\big(\textstyle\sum_j x_j(n - D_j^{\rightarrow} - D_i^{\leftarrow})\big)\Big)$$

- Let x^* be the equilibrium point of the system, then the linearized system model becomes

$$x_i(n) = \sum_j a_{ij} x_j(n - D_j^{\rightarrow} - D_i^{\leftarrow}) \qquad (*)$$

$$a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}^*}$$

Equivalent System Model

- Consider following linear system

$$x_i(n) = \sum_j a_{ij} x_j(n - D_i) \qquad (+)$$

- Lemma 1: System $(*)$ is stable under all heterogeneous directional delays $D_i^{\rightarrow}$ and $D_i^{\leftarrow}$ if and only if system $(+)$ is stable under all round-trip delays D_i
- System $(+)$ has a simpler shape than $(*)$, so we only consider stability of $(+)$ in the rest of the talk

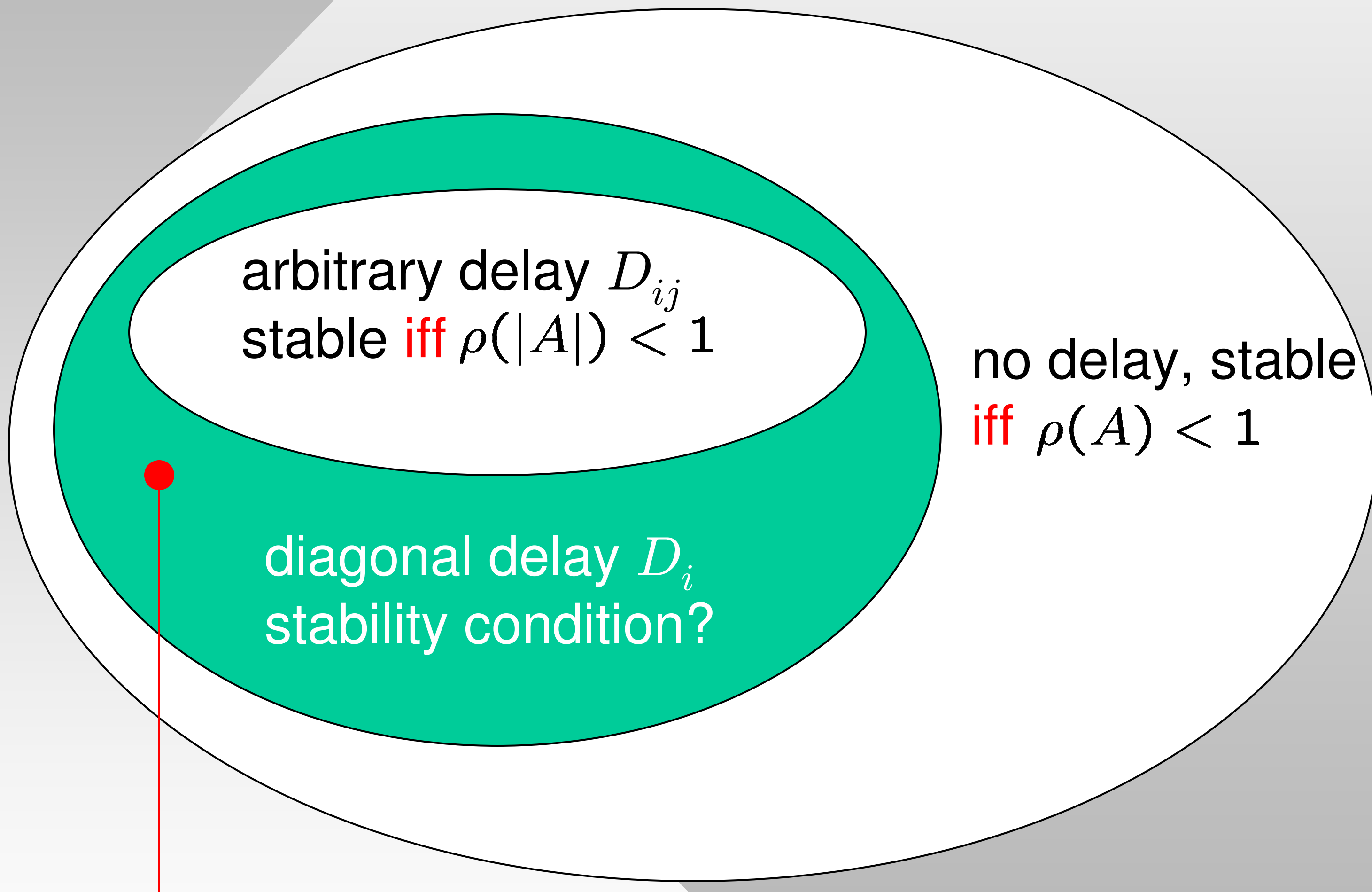
Current Stability Results I

- Assume that the Jacobian matrix A does not involve any delay
- Definition 1: We call a system stable independent of delay if its stability condition does not depend on delays
- Clearly, system $(+)$ under zero delay is stable if and only if $\rho(A) < 1$
- Consider (+) under arbitrary delay D_{ij}

$$x_i(n) = \sum_j a_{ij} x_j(n - D_{ij}),$$

which is proved to be stable if and only if $\rho(|A|) < 1$

Current Stability Results II



stable under diagonal delay
D_i but with $\rho(|A|) > 1$

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Induced Matrix Norm

- Definition 2: The induced matrix norm ||.|| of a given vector norm ||.|| is defined as follows:

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

- Examples
  - Spectral norm $||A||_2 = \sqrt{\rho(A^*A)}$ (where A^* is the conjugate transpose of A) is induced by the L^2 vector norm
  - Maximum absolute column sum norm $||A||_1 = \max_j \sum_{i=1}^{N} |a_{ij}|$ is induced by the L^1 vector norm
  - Maximum absolute row sum norm $||A||_1 = \max_i \sum_{j=1}^{N} |a_{ij}|$ is induced by the L^∞ vector norm

Extending A Previous Result

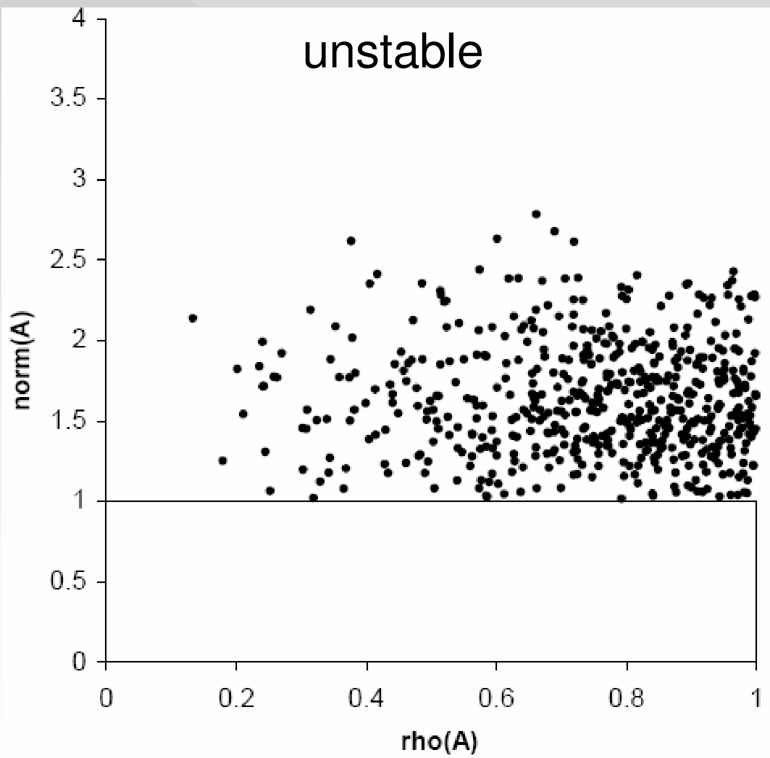
- It is proved by Zhang *et al.* (SIGCOMM04) system $(+)$ is stable if A is symmetric and $\rho(A) < 1$
- However, this result is very restrictive
- Utilizing induced matrix norms, we can obtain an alternative proof of this result and have the following observation
- Corollary 1: System $(+)$ is stable for all delays D_i if

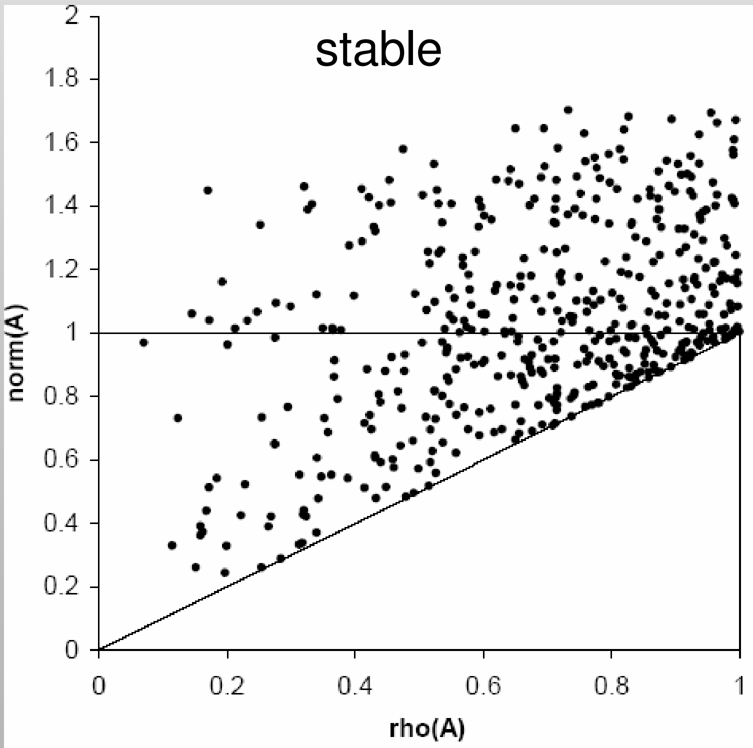
$$||A||_2 < 1$$

- Clearly, this condition is tighter (i.e., less restrictive) than the previous result

Verification of Corollary 1

- Matlab simulations
  - Generate 3000 random two-by-two matrices and plot (x,y) where $x = \rho(A)$ and $y = \|A\|_2$ of stable and unstable matrices on a 2-D plane





- Thus, Corollary 1 is a sufficient but not necessary

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Tighter Sufficient Conditions I

- Definition 3: A vector norm ||.|| is monotonic if for all x, y in R^n such that $|x| \leq |y|$,it follows $||x|| \leq ||y||$
- Theorem 1: If there exists a monotonic vector norm $||.||_\alpha$ such that the induced matrix norm $||A||_\alpha < 1$, system $(+)$ is stable regardless of delay D_i
- Monotonic norms can be generated using the following result
- Theorem 2: Matrix norm $||A||_2^w = ||WAW^{-1}||_2$ for any non-singular diagonal matrix $W =$ diag(w) is a monotonic induced matrix norm

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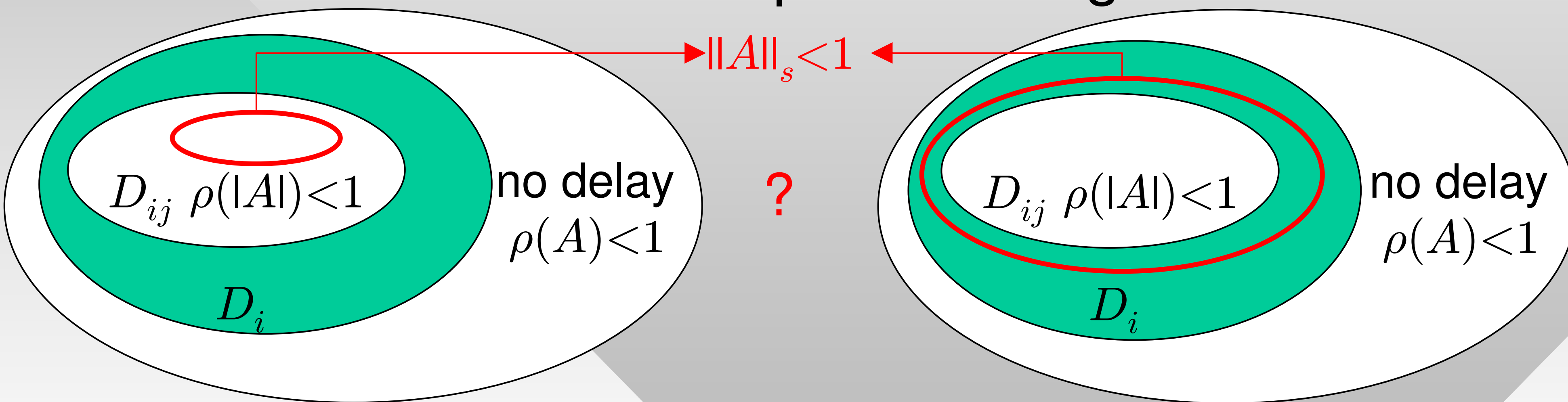
Tighter Sufficient Conditions II

- Corollary 2: System $(+)$ is stable for all delays D_i if

$$||A||_s = \inf_{W \in \mathcal{P}^*} ||WAW^{-1}||_2 < 1$$

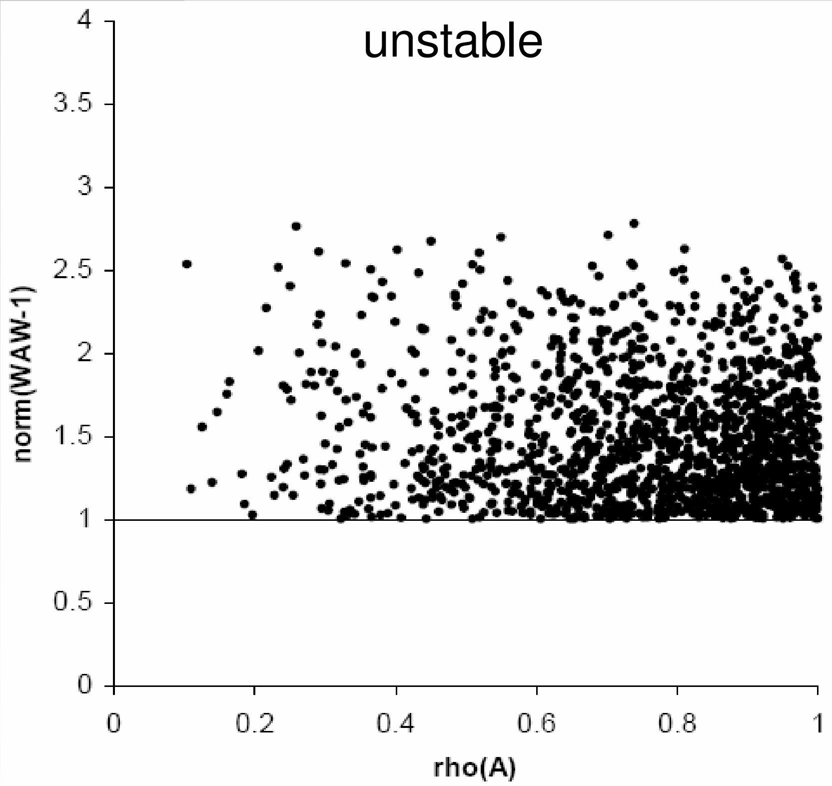
where $\mathcal{P}^*$ is the set of all positive diagonal matrices

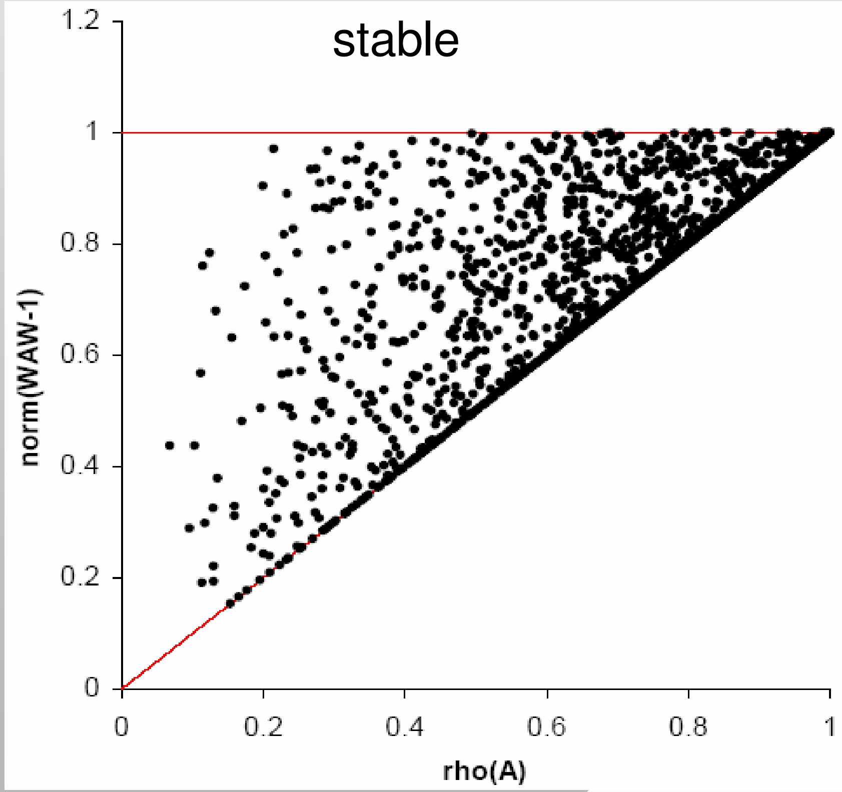


- Theorem 3: For any matrix A, we have $||A||_s \le \rho(|A|)$
- Therefore, Corollary 2 does not hold for arbitrary D_{ij}. But, is it tight for D_i?

Verification of Corollary 2

- We next use Matlab simulations to verify Corollary 2
  - Generate 10000 random two-by-two matrices and plot 3535 stable/unstable matrices





- Conjecture: Condition in Corollary 2 is both sufficient and necessary for $(+)$ to be stable under any delay D_i

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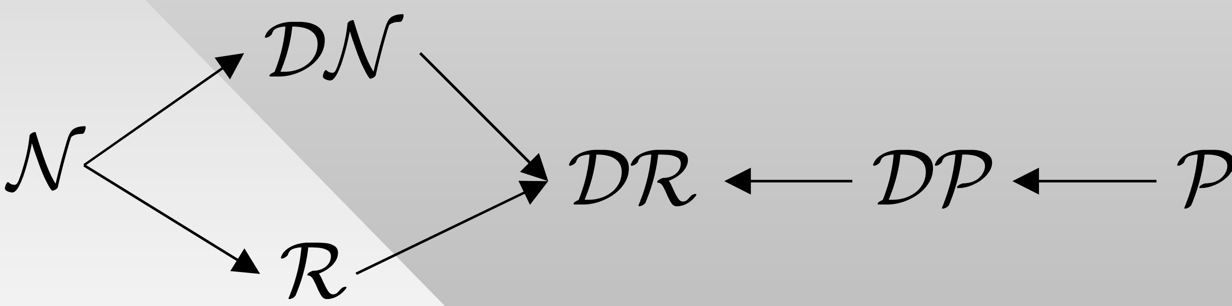
Delay-Independent Stable Matrices I

- $\mathcal{N}$: normal matrices, A is in $\mathcal{N}$ if $AA^* = A^*A$
- Definition 4: Matrix A is diagonally similar to B if there exists a non-singular diagonal matrix W such that $WAW^{-1} = B$
- $\mathcal{DN}$: the set of matrices that are diagonally similar to $\mathcal{N}$
- $\mathcal{P}$: the set of non-negative/non-positive matrices
- $\mathcal{DP}$: the set of matrices that are diagonally similar to $\mathcal{P}$
- $\mathcal{R}$: the set of radial matrices, A is in $\mathcal{R}$ if $\|A\|_2 = \rho(A)$
- $\mathcal{DR}$: the set of matrices that are diagonal similar to $\mathcal{R}$

Delay-Independent Stable Matrices II

- Theorem 4: The following matrices are stable under all diagonal delay D_i if and only if $\rho(A) < 1$: $\mathcal{N}$, $\mathcal{DN}$, $\mathcal{P}$, $\mathcal{DP}$, $\mathcal{R}$, $\mathcal{DR}$

- These matrix classes satisfy the following relationship, where $A \rightarrow B$ denotes $A \subset B$



- $\mathcal{DR}$ is the largest class of matrices that are stable under all diagonal delay D_i if and only if $\rho(A) < 1$

Application to Max-min Kelly Control

- MKC end-user equation:

constant

$$x_i(n) = (1 - \beta p_i(n - D_i^{\leftarrow}))x_i(n - D_i) + \alpha$$

sending rate of user i feedback $p_i(n) = g\left(\sum_j x_j(n - D_j^{\rightarrow})\right)$

- Stability of MKC under homogeneous parameters α and β has been proved

- Theorem 5: Single-link MKC with heterogeneous α_i and β_i is stable under all diagonal delay D_i if

$$0 < \beta_i(p^* + \sum_{i=1}^{N} x_i^* p') < 2, \quad i = 1, \ldots, N$$

where $x_i{}^*$ and p^* are stationary points of $x_i(n)$ and $p(n)$

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Wrap-up

- In this paper, we studied stability of max-min congestion control systems under diagonal delays
- Our results improved the understanding of delay-independent stability from the requirement that $\rho(A) < 1$ and A is symmetric to the simple condition that $||A||_s < 1$
- Simulations suggest that $||A||_s < 1$ is also a necessary condition
- The obtained results are of broader interest and apply to any system that can be modeled by $(+)$ or $(*)$